tarian value—and the sponsors of the present movement are among those who believe that it has—that value is bound up with some degree of continuity in its teaching. Much of its importance lies in the fact that it is a part of the common background and experience of educated men. The Pythagorean theorem is significant not merely because it is true, but because intelligent people everywhere know that it is true. Changes in method and content must not be so radical as to make one man's geometry unrecognizably different from another's. The skill acquired by experienced teachers in years of service is not to be redirected too abruptly. Those colleges whose avowed purpose is to attract the best pupils from small and remote schools must continue to recognize the traditional methods which will prevail in those schools for a long time to come.

Although it is not the function of this committee to formulate detailed recommendations, it may be appropriate to set down some further considerations which bear on the working out of the problem.

A most essential feature of the Board's examinations is the presence of really substantial "originals." The power exhibited by the better candidates in attacking such originals bears eloquent testimony to the ability of the candidates and to the effectiveness of the instruction that they have received. Unless the revision can be carried through without detriment to this feature of the requirement, its purpose will have been defeated at the outset. If it were demanded that the range of the course should be largely increased without diminution of its average depth, the attempt might be dismissed as obviously hopeless. But if it is granted that the maximum depth rather than the average depth is the important thing, the task may not be impossible. Specifically, it may be recalled that originals to try the powers of any candidate can be based on the content of the first three books of plane geometry.

If the pupil can once be brought to appreciate what consecutive deductive reasoning means, he will have a healthier attitude toward geometry if he then learns more facts than are completely proved, with frank and explicit recognition of logical omissions, than if he tries to believe that the last word has been said on the subject when the last page of the book is reached. To admit that geometry as we actually know it, personally and individually, is far from abstract finality, is not to surrender its essential character, but to bring it closer to the rest of human experience.

The detailed choice of material will naturally be a subject for extensive study and debate. There will be differences of opinion as to the amount of stress to be placed on the content of the first book of solid geometry, according as it is considered to be the most important or the most repellent part of the subject. A distinction may perhaps be made between the numerous and elusive proofs whose precise form is significant only in connection with a particular order of propositions, and the few characteristic and substantial demonstrations which erect the first three-dimensional structures from two-dimensional elements. The measurement of solid figures will undoubtedly be included, but is susceptible of treatment varying all the way from theoretical prominence to incidental

computation. The geometry of the surface of the sphere may be regarded as of prime importance in a science which still goes by the name of earth-measurement, or as comparatively inaccessible. A no less fundamental problem will be the selection of the parts of plane geometry to be omitted.

The amount of ground that can be covered is not to be estimated by a mere count of propositions. Much time and maturity of experience are required for the assimilation of fundamental concepts, which when once mastered appear so obvious that it is hard to make due allowance for the process of their introduction. Much may be accomplished by informal and repeated reference to the concepts of solid geometry early in the course, so that they are long since familiar when they first appear in the formal order of propositions. Or it may be that intuitive geometry in the junior high school is an essential foundation for the satisfactory carrying out of the whole program.

It is not to be overlooked, on the other hand, that an integral part of the project is the formulation of a second course for the senior high school, to be articulated with the first and to include the parts of plane and solid geometry not covered by the first year's work. While not a matter of concern to the numerical majority who take only one year of demonstrative geometry, adequate provision for the scientifically and professionally important minority who go further is a vitally essential part of the undertaking.

Various suggestions have been made for utilizing this opportunity of bringing about a closer coordination between geometry and other branches of mathematics. It has been urged, for example, that more extensive use should be made of algebraic methods. On the other hand, it may be worth while to point out that the "paradox" of extraneous roots in algebra is merely a matter of failing to distinguish between a proposition and its converse. It has been proposed that numerical trigonometry should be included in the geometry course as well as in the course in algebra. The work in geometry may both profit by the use of coordinates and clarify the pupil's understanding of them if he encounters them elsewhere. Without any attempt at a premature introduction of methods of the calculus, some hints as to the nature and efficacy of those methods may be left where the intelligent student will find them.

The working out of these suggestions is largely a matter for teachers and textbook writers rather than for the framers of formal entrance requirements. It may be possible nevertheless for the latter also to exert some influence toward putting them into effect, without impairing the distinctive character of the course in geometry as such.

A necessary auxiliary to the satisfactory establishment of the new courses will be the preparation of suitable textbooks. It is to be anticipated, however, that such textbooks will be forthcoming, if it becomes clear that the attitude of schools and colleges is such as to render them commercially feasible.

As a means of putting into effect the proposals under discussion, *the present committee suggests that a small committee, of three or five members, be appointed by the College Entrance Examination Board, to formulate appropriate requirements*

for tentative adoption as alternatives to the present Mathematics C and Mathematics D. A small committee is recommended because the speedy and vigorous adoption of a temporary working basis is regarded as more important than an attempt to reconcile all varieties of opinion in the present transitional stage. The committee would naturally be authorized and expected to avail itself of the advice of persons outside its membership. It is further suggested that the same committee be asked to construct examination papers under the new requirements for a period of years, subject to suitable revision and control, instead of calling upon the regular examiners to assume this additional responsibility, and that it be regarded accordingly as a standing committee to observe the progress of the experiment and to make subsequent recommendations as occasion demands.

Respectfully submitted,

Gertrude E. Allen	J. O. Hassler
C. M. Austin	Dunham Jackson (Chairman)
Ralph Beatley	C. N. Moore
Walter F. Downey	W. D. Reeve
Elizabeth L. Hall	Edwin W. Schreiber

RANDOM SAMPLING[1]

By W. A. SHEWHART, Bell Telephone Laboratories, Inc.

I. INTRODUCTION

Have you ever noticed a number of small boys playing in a vacant lot when one of the gang comes along with something new? Immediately there is an outburst of questions—"What you got?" "What's it good for?" This afternoon I am thinking of a somewhat similar scene. Instead of a group of small boys we have a host of research workers and industrial artisans worrying over the interpretation of chance variations in samples of data which they cannot explain. Into that group comes another from a strange land with a new tool—the theory of random sampling for use in interpreting chance phenomena. Immediately we hear the same two questions: "What you got?" "What's it good for?" What I have in mind to say this afternoon has to do with the answers to these two questions. To begin, however, let us look at the practical and almost universal need for such a tool, because what I have to say is tempered by such needs.

II. NEED FOR A THEORY OF SAMPLING

1. *Universal Need for Sampling Theory.*

The ultimate aim of research is to reduce everything to known laws, thereby doing away with chance. In the laboratory where all but one variable can often be quite thoroughly controlled, research progress has been very encouraging.

[1] A paper presented at the invitation of the program committee of the Mathematical Association of America at its meeting at Brown University, Providence, Rhode Island, September 8, 1930.

Even here, however, when the most refined measurements are made, such, for example, as the determination of the charge on an electron by Millikan, there remains a nucleus of effects of uncontrolled or chance causes. In other words, chance we have with us always.

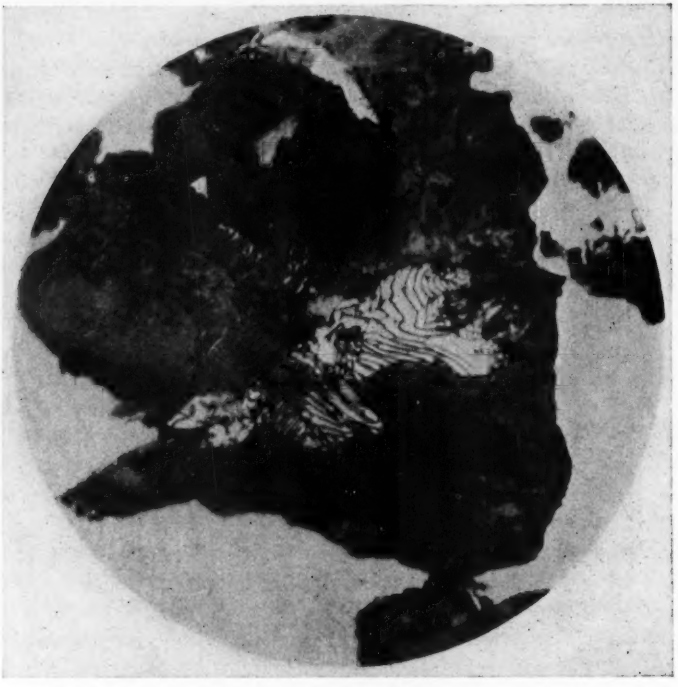

FIG. 1

Bridgman in *The Logic of Modern Physics* admirably sums up the state of affairs thus:

> A situation like this merely means that those details which determine the future in terms of the past may be so deep in the structure that at present we have no immediate experimental knowledge of them and we may for the present be compelled to give a treatment from a statistical point of view based on considerations of probability.

Even the field of "exact" science, so-called, is being invaded by statistical concepts. No longer do we think of physical properties as constant quantities—instead, they are *frequency distribution functions*. In the same way, we think of relationships between physical quantities as frequency distributions in two or more dimensions. For these reasons it appears that all prediction must be based

upon samples and that *prediction within limits* is the only kind possible. The need for sampling theory appears to be universal.

To emphasize the statistical nature of properties of materials still often treated as constants, let us look through a microscope at a piece of ordinary steel, Fig. 1. What we see is anything but a homogeneous, isotropic body. Why this heterogeneous structure? The answer is—it is attributable to the effects of chance or unknown causes.

What is the effect of such irregularities upon the physical properties of steel when produced in some useful form as, for example, a supporting strand, a piece of which is shown in Fig. 2? The answer is that a physical property, say the

FIG. 2

breaking load, of such strand will be distributed as indicated in Fig. 3.

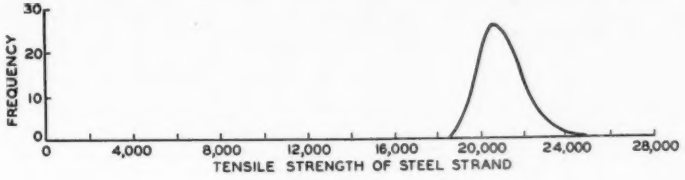

FIG. 3

Now, let us look at a cross section of another important structural material —wood, Fig. 4. This time we do not need a microscope to see the effects of chance causes upon the structure of the material.

Fig. 5 shows roughly what such irregularities do to the modulus of rupture of four kinds of telephone poles. Note the wide spreads of these distributions as compared with their means.

2. *Typical Specific Needs.*

We have said enough about the general need for a theory of sampling. Let us now consider some specific typical illustrations from the field of engineering and physical research, although similar ones could have been taken from any other field of human endeavor.

A. *Protection Against Loss of Life or Property.*

Two classes of conditions arise where an engineer must use sampling theory

in protecting against loss of life. One is where the thing upon which life depends cannot be measured otherwise than by sampling. The other is where the thing cannot be measured except through destructive tests.

FIG. 4

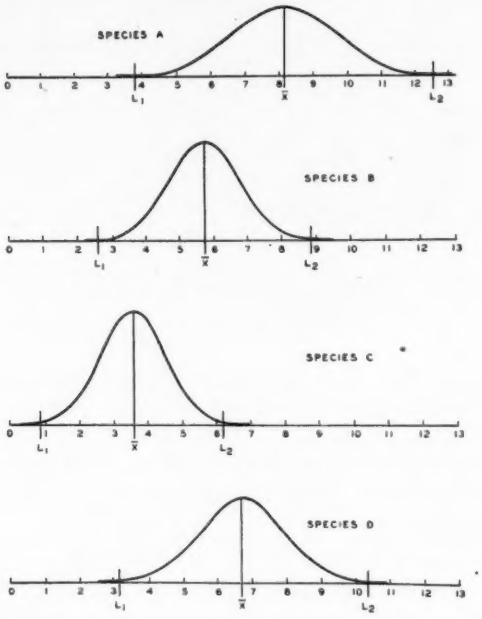

FIG. 5

As an example of the first class a civil engineer wants to build a dam or levee to hold back the flood waters of a given area, and hence wants to know the maximum run-off of that area to be expected in the future. If he does not build his

structure to take care of this maximum load, loss of life and property may ensue. From a sample of data such as that in Table 1 together with any other pertinent information, he must derive an estimate of the maximum run-off to be expected.

TABLE 1

Average Weekly Run-off—cu. ft. per second per sq. mile

Run-Off	Freq.	Run-Off	Freq.	Run-Off	Freq.	Run-Off	Freq.
29.30	1	6.82	1	4.27	8	1.85	57
15.50	1	6.77	2	4.16	7	1.74	60
13.60	1	6.52	2	4.05	5	1.64	50
12.65	1	6.45	2	3.95	10	1.53	45
12.50	1	6.36	2	3.85	9	1.43	50
11.55	1	6.25	2	3.75	10	1.32	50
11.30	1	6.06	2	3.64	10	1.21	57
10.52	1	5.95	4	3.54	9	1.10	43
10.35	1	5.85	2	3.43	15	1.03	15
9.90	1	5.74	1	3.32	17	.97	20
9.70	1	5.64	3	3.21	15	.92	13
9.65	1	5.53	1	3.10	17	.87	15
9.40	1	5.43	3	3.00	17	.82	15
9.33	1	5.32	2	2.90	20	.76	11
8.82	1	5.22	1	2.80	22	.71	8
8.60	1	5.11	4	2.69	26	.66	7
8.43	1	5.00	4	2.59	20	.61	2
8.05	1	4.90	3	2.48	30	.55	3
7.50	1	4.79	2	2.37	23	.50	2
7.45	1	4.69	5	2.26	30	.40	1
7.30	1	4.58	2	2.16	33	.34	3
7.23	1	4.47	9	2.06	35	.29	3
7.16	1	4.37	4	1.95	40		

The following typical questions are suggestive of examples of the second class. What is the breaking load of any structural member such as the steering-rod in your car? How long may you safely use any physical thing which deteriorates with age, such as the rubber in the tires of your car? Will your fire extinguisher work? At what pressure will the boiler in the basement of your home explode?

As a specific illustration, what can we say about the quality of the clay conduit piled up in the background of Fig. 6 from the results of the breaking tests conducted on the sample of four pieces shown in the foreground of this figure?

B. *How Shall we Take a Sample?*

How would you select a sample of 100 poles from those in the pole-yard of Fig. 7 to be inspected for modulus of rupture?

As another illustrative question, what plan would you propose for sampling the quality of the soldered terminals in a telephone office. To be more specific,

let us consider the panel in Fig. 8 on which there are 4500 such terminals. In a large office there are about 50 such panels. Of course we must keep in mind that the very process of inspecting the terminals may tend to loosen the soldered connections and that the taking of a second sample, if the first is found to be too small, is an expensive procedure.

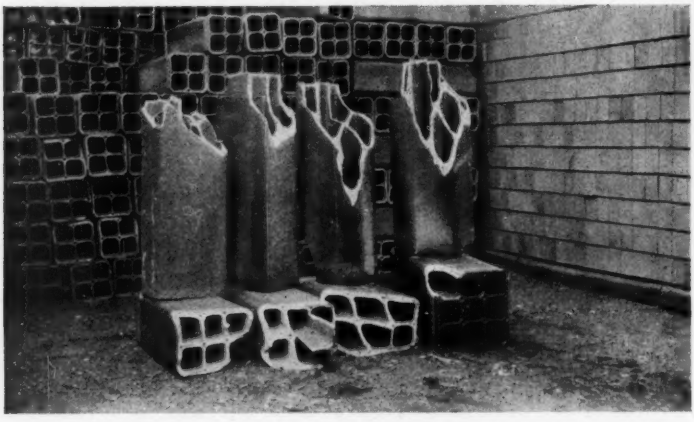

FIG. 6

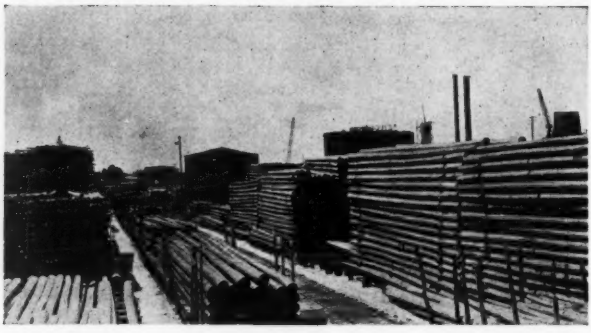

FIG. 7

In a similar way we may ask: How would you sample the bricks from a kiln, a sheet of brass, a keg of nails, a boat load of hides, a shipment of 50 barrels of oil, a barrel of flour, your drinking water, etc.? We might extend at great length the list of such problems.

Furthermore, the question as to how large a sample shall be taken enters into

every sampling problem. In fact, we need to know how to take a sample and how many to take in almost everything we do.

C. *How Shall We Specify the Standard of Quality?*

Typical questions are: How shall we estimate and tabulate the physical and chemical properties of materials in such a way that they may be used to greatest advantage in satisfying human wants, remembering that such qualities are al-

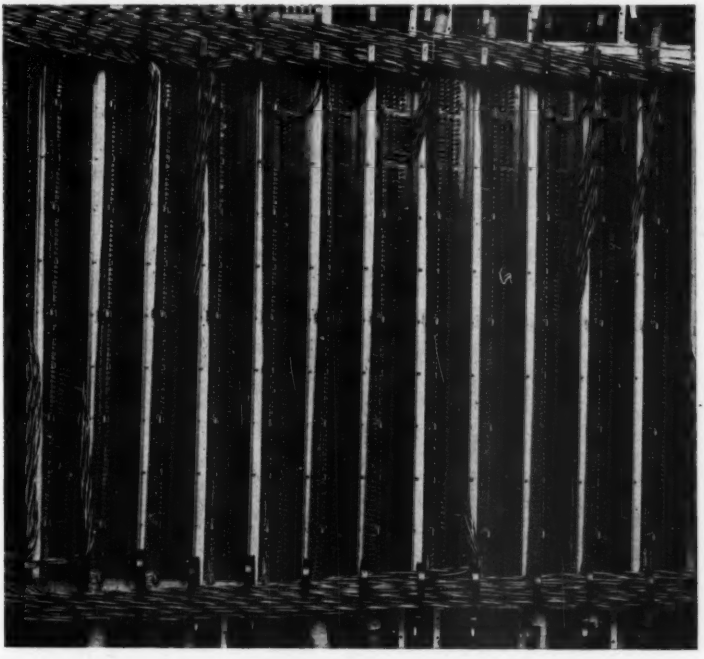

FIG. 8

ways frequency distribution functions? How shall we find such functions from the available sample of data? Shall we bother to try to specify the functional form or will it be sufficient merely to specify certain characteristics of a distribution function such as the average and standard deviation? How much information of this character is it necessary for a design engineer to have in order to secure minimum variability in the qualities of a designed structure at a given cost?

The standard qualities of the things we make and the things we eat to sustain life are frequency distributions. How shall we specify these to make it possible for inspection to give the greatest assurance of satisfactory quality?

D. *When Must Variability Be Left to Chance?*

When is the difference in quality of material from two or more sources greater than should be left to chance? For example, are the monthly variations

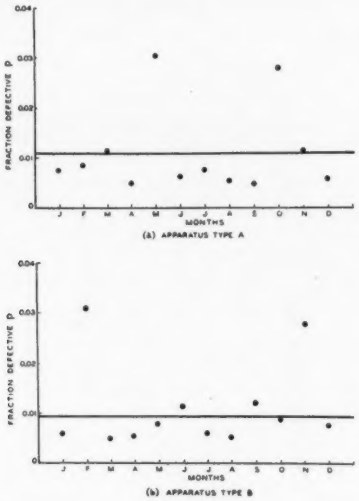

FIG. 9

TABLE 2

5045	4565	4895	4335	4550	4685	4850	4700	4500
4350	4410	4255	5000	4700	4430	4450	4500	4765
4350	4065	4170	4615	4310	4300	3635	4840	4500
3975	4565	3850	4215	4310	4690	3635	5075	4500
4290	5190	4445	4275	5000	4560	3635	5000	4850
4430	4725	4650	4275	4575	3975	3900	4770	4930
4485	4640	4170	5000	4700	2965	4340	4570	4700
4285	4640	4255	4615	4430	4080	4340	4925	4890
3980	4895	4170	4735	4850	4080	3665	4775	4625
3925	4790	4375	4215	4850	4425	3775	5075	4425
3645	4845	4175	4700	4570	4300	5000	4925	4135
3760	4700	4550	4700	4570	4430	4850	5075	4190
3300	4600	4450	4700	4855	4840	4775	4925	4080
3685	4110	2855	4700	4160	4840	4500	5250	3690
3463	4410	2920	4095	4325	4310	4770	4915	5050
5200	4180	4375	4095	4125	4185	4500	5600	4625
5100	4790	4375	3940	4100	4570	4770	5075	5150
4635	4790	4355	3700	4340	4700	5150	4450	5250
5100	4340	4090	3650	4575	4440	4850	4215	5000
5450	4895	5000	4445	3875	4850	4700	4325	5000
4635	5750	4335	4000	4050	4125	5000	4665	
4720	4740	5000	4845	4050	4450	5000	4615	
4810	5000	4640	5000	4685	4450	5000	4615	

in qualities of the two types of product shown in Fig. 9 indicative of the presence of assignable causes of variation or those which should not be left to chance?

Or again suppose an industrial research engineer interested in the production of a high resistance material of a given type finds that the first 204 successive pieces gave the results shown in Table 2 expressed in megohms. Suppose that this engineer asks if he has gone as far as he can reasonably expect to go in eliminating variability. What does sampling theory enable him to say?

E. *When Do We Know that Data are Good?*

If one gets to talking with any group of engineers about the application of sampling theory to their problem, he will likely run up against a criticism like the following:

"I agree that statistical or sampling theory has its place, perhaps, if we start with good data such as Millikan's measurements of the charge on an electron. In that case some of the high-brow theory may help one to get what he believes to be the most satisfactory estimate of a probable error. However, suppose that one starts with almost any ordinary set of engineering data as, for example, the observed values of modulus of rupture of 58 telephone poles, Table 3. Do you mean to tell me that a statistician claims to be justified in using all his high-brow formulas on this set of data to find the expected value and standard deviation of the modulus of rupture to the same degree that he would be in using the same formulas to find the best estimate of the charge on an electron and the probable error of his estimate?"

TABLE 3

Modulus of Rupture in lbs. per sq. in. of Type A Telephone Poles.

6930	6700	5910	7290	5090	6830
7100	5270	6370	5880	6870	7800
6650	6830	7330	5310	8120	6640
4690	7540	6140	6040	6650	9360
7550	7620	5510	8230	5550	6890
7620	6770	6560	4580	7720	9360
5990	7630	4880	5400	6770	5090
6970	4690	6010	5560	8220	5760
6490	5980	4580	5480	7600	
7630	8230	4610	6930	8190	

This kind of question is likely to give a statistician quite a jolt when he hears it for the first time. All of us know that when a man of research gets as good a set of data as that of Millikan he doesn't fuss much with the statistical theory of estimation. We also know that there are many sets of data not worth the paper they are written on—certainly not worth fussing over with high-brow instruments of estimation.

If sampling theory is in a position where those who have good data don't need it and those who do not have good data can't use it, *the industrial statistician might just as well shut up shop and find a new interest in life.* Discussions of sampling theory should show the way out of this apparent dilemma.

3. *Résumé of Situation to be Met by Theory.*

We have seen the kinds of questions that confront the theory of random sampling. In general, the theory must assist us in interpreting a sample in terms of the future under practical conditions where the variations in the sample are produced by unknown or chance causes. It is just this thing that the theory can be made to do for the practical man.

I fear, however, that great progress in application will not come until we have a comprehensive treatise of the theory in which emphasis is laid upon some of the essential elements of the theory often completely omitted from available discussions. The theory should be made to rest upon assumed physical laws and yet such laws are often not mentioned; the theory should be made broad enough to include practical problems and yet we shall soon see that certain current definitions of random sample exclude this possibility; some of the very important theorems of specification such as those of Tchebycheff and Camp-Meidell should be made available even though to do so pushes out some of the high-brow specifications of frequency curves of so much less practical importance. I fear that it is not sufficient merely to inform the practitioner that the theory is to be used as a tool. Instead, it is necessary to show him at what point the theory becomes of use and to show him what the theory will do *that he cannot hope to do as well without it.* The theory of random sampling *can be made* to meet this situation as we shall now see.

III. THE THEORY OF RANDOM SAMPLING—WHAT IS IT?

1. *What is a Random Sample?*

After I had decided what I wanted to say, it occurred to me that it might be well to see if the proposed remarks had to do with the subject assigned for discussion, namely, *Random Sampling.* Thereupon I gathered about me a goodly number of books on the theory of probability and the theory of statistics, and tried to find out the generally accepted meaning of random sampling. I thought it interesting to start by finding out what is meant by a sample. Two unabridged dictionaries gave the following definitions:

1. "A sample is a portion taken at random out of the quantity supposed to be homogeneous so that the qualities found in the sample may reasonably be expected to be found in the whole."[1]

2. "A sample is a part of anything taken at random out of a large quantity and presented for inspection or intended to be shown as evidence of the quality of the whole."[2]

If one accepts either of these definitions, it is obvious that the subject of my discussion is tautological. It should have been *Sampling.* To accept either of these definitions would play havoc with what I had in mind to say. If either of these definitions were accepted, much of what is written in the theory of sampling would have to be revised; because in practically every discussion we find the author talking about samples that are not random, a thing which obviously could not exist if, in accord with either of the definitions, a sample is

[1] Funk & Wagnall's Unabridged Dictionary.

[2] Century Dictionary and Encyclopedia.

something that is random. Is it any wonder that many of the possible applications of sampling theory go unmade?

The more I read, the more uncertain I became as to the prevalent conception of random sample. One thing certain, however, is that many of those found in standard places are not of much interest to a practical man.

For example, Yule, in that treasure house for statisticians, *An Introduction to the Theory of Statistics*, indicates that the usual concept of random sample is one drawn with replacement, although he explicitly rejects the term *random* because he takes it to mean simply *haphazard*. Jones also would have us believe that a random sample is one drawn with replacement. For example, he says in effect: To select 99 sheep from 999, number each sheep and place in a box 999 tickets numbered 1 to 999, one to correspond to each sheep, then pick out 99 tickets in succession being careful to replace each and shake up the box before picking out the next; if there were absolutely no difference between the tickets, such as would cause one to be picked more easily than another, the selection made in this way would be random.

Now, if a random sample were only that kind of a sample and if the theory of random sampling had to start with that kind of a sample, you can imagine how enthusiastic a purchaser of 999 sheep would be about the theory. To such a man that method of sampling would be foolish.

Not only is such a method of sampling often foolish from a practical viewpoint—very often indeed it is impossible to sample in this way. How would you make this kind of a random test of the tensile strength of a coil of wire—Fig. 10? I can't.

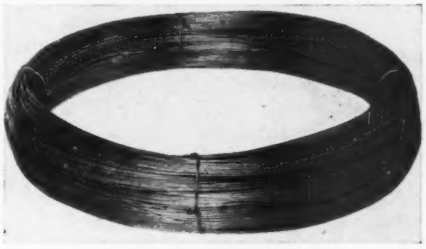

FIG. 10

Such a concept of random is certainly too narrow to get very far with in practice. The sample described by Yule is random, of course, but so are certain other kinds of samples as we shall soon see.

To start with, however, we must do as the physicist does in formulating a theory—adopt certain physical hypotheses or laws. This will give us a basis for a general definition.

2. *Law of Large Numbers.*

In the simplest form this law can be stated as follows: If an event which can

happen in two different ways be repeated a number n times under the same essential conditions, the ratio p of the number of times that it happens in one way to the total number n of trials will approach a definite statistical limit p' as the number n is increased indefinitely. Formally this law may be written

$$\lim_{n\to\infty} p = p',$$

where $\lim_s$ stands for the statistical or stochastic limit.

A slightly more extended form of this law is as follows: If we make a series of n measurements

$$X_1, X_2, \cdots, X_i, \cdots, X_n$$

of some quality characteristic X in a way such that each measurement is made under the same essential conditions, the ratio p of the number of times that an observed value X will be found to lie within any specified range X_r to X_s to the total number n will approach a statistical limit p' as the number n is increased indefinitely.

An even more general statement of this law is: If we take a series of m samples of n measurements

$$\begin{array}{l} X_{11}, X_{12}, \cdots, X_{1i}, \cdots, X_{1n}, \\ X_{21}, X_{22}, \cdots, X_{2i}, \cdots, X_{2n}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ X_{m1}, X_{m2}, \cdots, X_{mi}, \cdots, X_{mn}, \end{array}$$

in such a way that each one of the m samples is drawn under the same essential conditions, and if we let θ be a symmetric function or statistic of the n values of X in a sample of size n, the ratio p of the number of times that the observed value of θ will be found to lie within the range θ_1 to θ_2 to the total number m of samples will approach a definite statistical limit p' as the number m of samples is increased indefinitely.

No doubt this law can be generalized even further, but in this form it serves the purposes of the present discussion.

We shall now assume without any consideration of the fine points involved—adequately discussed in so many places—that the statistical limit p' is an objective probability that can be substituted for the *a priori* probability in the mathematical formulas of the theory of specification, distribution, and estimation soon to be considered.

Of course, the Law of Large Numbers is perhaps the most universal law of nature. Even the parameters in the so-called exact laws are apparently constant only to the degree to be expected of the statistical limits which they really are.

3. *Definition of Random Sample.*

A sample drawn under conditions such that the Law of Large Numbers applies will be termed a random sample.

Obviously a random sample thus defined may be one drawn with replacement as characterized by Yule and Jones; one drawn without replacement, or one in which any part of the sample is drawn with replacement and the remainder without. So also is a Poisson sample a random sample, not to mention others that come under our general definition.

4. *Theory of Random Sampling.*

The statement of the methods of applying the Law of Large Numbers in the interpretation of random samples in terms of the future will be termed the theory of random sampling. At least five specific elements in this theory must be considered. They are:

1. The Law of Large Numbers and physical postulates making this law useful.
2. The mathematical theory of specification.
3. The mathematical theory of distribution.
4. The theory of estimation.
5. The rôle of human judgment.

We shall now see how these elements function in the practice of the theory.

IV. WHAT IS THE THEORY GOOD FOR?

1. *What do We Want the Theory to Do?*

In Part II we considered several types of problems. It will be helpful to generalize these problems still further by reducing them to two types. We may do this by developing a theory which will do two things.

a. Establish rules for prediction when observed data have been obtained under the same essential conditions.

b. Establish criteria to assist an experimentalist in determining when his data have been taken under the same essential conditions.

Suppose we note how the specific types of questions A, B, C, D, and E, of Part II can be reduced to one or the other of these two fundamental problems.

To predict such things as the maximum run-off of a flood area or the breaking load of any structural member, to set up standards of quality, and to determine how large a sample to take, involve either directly or indirectly the problem of going from a sample of n data to a set of m similar data not yet taken.

The Law of Large Numbers tells us that we can take this step *provided* the measurements have been made under the same essential conditions. So far as we know, however, there is no available basis for taking this step unless the samples are random in the sense of the generalized definition of Part III. Questions as to how we shall take a sample and what good data are, therefore, resolve themselves into the problem of taking data under the same essential conditions insofar as the effects of chance causes are concerned. So also, the very important problem of determining when variability in the characteristic of the phenomenon must be left to chance reduces to the second general problem stated at the beginning of this section.

In this same way, it appears that we can reduce any question involving the interpretation of a sample to one or the other of these two fundamental problems.

Hence, if we can show that the theory of random sampling helps in solving these two problems, we will have shown that it has in it something useful in the interpretation of samples.

2. *Contribution of the Theory of Random Sampling to the Solution of the First Fundamental Problem.*

It follows from what has been said that this problem can be considered as that of going from a sample to what we ordinarily term a universe, as schematically illustrated in Fig. 11. This is also the problem of estimation, so-called by R. A. Fisher.

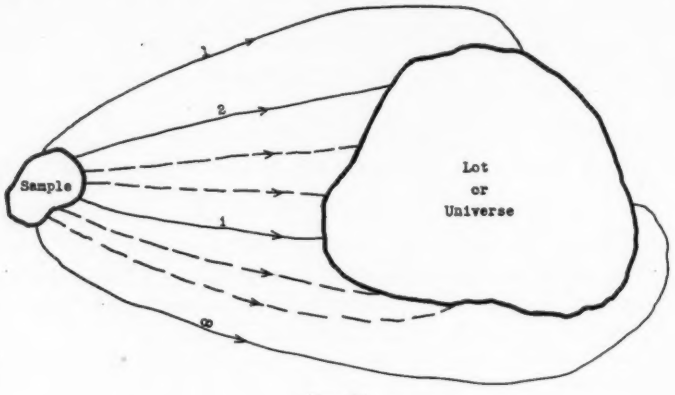

FIG. 11

There are in the literature the following three general different methods of going from a sample to its universe:

a. The *a posteriori* method.
b. The method of maximum likelihood.
c. The empirical method.

To mention these three in the same breath in the presence of a group of statisticians is almost certain to start an argument, for, as is well known, there is a wide divergence of opinion as to the comparative validities of these methods.

This afternoon, however, we have no desire to start such an argument. The only thing that we are interested in doing, is to call attention to the fact that we are apparently in a position to understand the nature of the problem better than ever before, and to state, as we shall soon do, certain specific things which seem to be true *independent* of the method we choose.

The *a posteriori* method is, of course, tied up with the name of Bayes and the discussions of probability of causes. The method of maximum likelihood is

largely the outgrowth of the work of R. A. Fisher, in particular, his paper[1] in the *Philosophical Transactions*, 1922, "On the Mathematical Foundations of Theoretical Statistics." The empirical method, so-called for the want of a better name, is that which involves the use of some constant c by which to multiply the observed value of a statistic θ in a sample of size n in order to get the best estimate of this same statistic of the universe.

One of the most interesting things from a practical viewpoint is that no one of these methods gives a unique answer which does not depend upon certain assumptions as to the nature of the universe. There are an indefinitely large number of such assumptions that may be used in connection with any one of the three generalized methods, and, depending on the assumption we wish to make, we get, through the use of available theory, a definite path of going from the sample to the universe as represented schematically by the lines connecting these two entities, Fig. 11.

Of course the practical man must choose *one* of the infinite set of possible ways of going from the sample to the universe. His choice is a matter of judgment. However, judgment can be exercized best after he becomes acquainted with available information contained in the detailed accounts of the significance of the choice of path.

Out of this theory there come four *fundamental contributions* of interest to the practical man, in spite of the fact that the formal theory does not give a unique answer to the fundamental problem.

1. The theory helps us to settle the right sort of difficulties and to raise the right sort of ulterior questions.

2. Through the use of judgment and the application of the Tchebycheff inequality, it follows that for most practical purposes it is sufficient to try to estimate only the average $\overline{X}'$ and the standard deviation σ' of the universe, there being little advantage to be gained for most practical purposes in attempting to go further in the way of specification.

3. No matter what one of the known ways of going from the sample to the universe we choose in a given case, it is desirable to start wherever possible with a knowledge of the average $\overline{X}$ and standard deviation σ of the observed set of n data.

4. It is evident that, in general, the best estimate of the standard deviation σ' of the universe derivable from the observed standard deviation σ of the sample is some multiple c times the observed standard deviation where, irrespective of the method used, c is greater than unity.

As a specific illustration, this part of the theory of random sampling indicates that the set of observed values of modulus of rupture of telephone poles given in Table 3 can best be summarized in terms of the average and standard deviation upon the assumption that these poles came from a constant system of chance causes or a condition wherein the probability that the modulus of rupture

[1] This paper also sets forth clearly the nature of the problems of specification, distribution, and estimation.

of a given pole lying within a given range is the same as the probability that any other pole will lie within this range. Furthermore, the best estimate of the standard deviation of the universe—this class of poles—is most likely something greater than the observed standard deviation, our particular choice of estimate depending upon which one of the infinite number of paths we choose to take. Under any conditions, however, provided the number in the sample is greater than, let us say, twenty-five as it is in this case, the magnitude of the correction factor is for all practical purposes unity.

The third contribution is of particular importance in the reduction of data in that it indicates that more extensive use than customary should be made of the two statistics, the average $\overline{X}$ and standard deviation σ of a sample, and furthermore that the sample size n should be recorded.

It is, of course, also of interest to know in any given case how the investigator himself would go from the sample to the universe, or, in other words, it is of interest to know the investigator's best estimate of the average $\overline{X}'$ and standard deviation σ' of the universe. Obviously, this estimate of the investigator contains in it not only the element of the original data but also the human element put in through his own personal judgment in choosing the requisite estimates. However, for some time to come, engineers and scientists in general will most likely want to be in a position where they can, if they please, choose some other set of estimates than those of the investigator. To make this possible, as already said, the investigator must tabulate the average $\overline{X}$ and the standard deviation σ in addition to his estimates.

It must be kept in mind, of course, that this infinite set of paths for going from the sample to the universe all rest upon a common assumption, namely, that each observation in the sample has been taken under the same essential conditions as any other observation.[1] Of course, if we do not start with such a set of data, it is of interest to know that there does not appear to be any available generalized method for going from the sample to the universe. This is of importance in that it indicates the necessity of eliminating causes of variability other than those which must be left to chance before attempting to apply the theory of random sampling as outlined above.

This situation seriously limits the application of this particular part of the theory. I believe that it is this state of affairs which so often prompts the engineer and scientific worker to criticize the applicability of the theory of random sampling on the score previously noted that his data are not sufficiently good to start with. In spite of this, however, it is the discussion of this first fundamental problem that receives the greatest attention in the literature which the man interested in the application of the theory of random sampling is most likely to read.

We shall now see that this situation calling for criticisms practically disap-

[1] We may, of course, think of the data as being statistics of samples of size n in which the samples have been taken under the same essential conditions even though the observations in a sample were not.

pears when we proceed with the discussion of the contribution of the theory to the solution of the second fundamental problem which in practice really arises before the first one does.

3. *Contribution of the Theory of Random Sampling to the Solution of the Second Fundamental Problem.*

By analyzing a set of data, can we get any indication as to whether or not they arose under the same essential conditions? As a specific example, did the set of 204 observations of resistance previously given in Table 2 arise under the same essential conditions? The theory of random sampling which includes the element of human judgment does provide a useful test as to whether or not the data have arisen under the same essential conditions.

Assuming the existence of the Law of Large Numbers, it follows that, if samples of size n are taken under the same essential conditions, there exists a frequency distribution function, Fig. 12, for any given statistic θ such that the

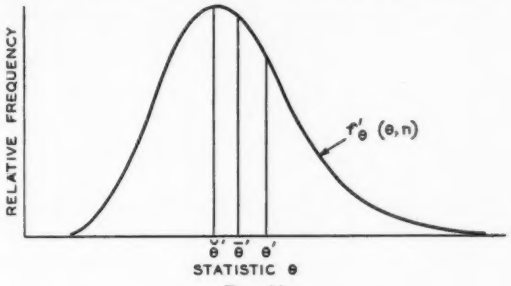

FIG. 12

integral of this function between any two limits θ_1 and θ_2 gives the probability that an observed value of θ will fall within these limits.[1] If, then, samples of size n do arise under the same essential conditions, and if this distribution function is known for these same essential conditions, we can say that P' of the observed values of the statistic θ for samples of size n should fall within the range fixed by the two limits $\bar{\theta}' \pm t\sigma'_\theta$ where $\bar{\theta}'$ is the expected or average value of θ and σ'_θ is the standard deviation of θ for samples of size n. The theory of specification and in particular Tchebycheff's theorem tells us that, if we take some range corresponding to the limits $\bar{\theta}' \pm t\sigma'_\theta$, then, no matter what the functional[2] form of the distribution, the probability of the value of the statistic θ falling within this range is greater than $1-t^{-2}$, where t is greater than unity. Thus Tchebycheff's theorem focuses our attention upon two characteristics of the frequency distribution function, namely, the average $\bar{\theta}'$ and the standard deviation σ'_θ. Knowing these, we can establish limits such as the dotted ones in Fig. 13 within which at least 89 per cent (if $t=3$) of the observed values of the statistic should lie provided the samples of size n arose under the same essential conditions.

[1] For our present purpose we shall assume that this distribution function is continuous.

[2] Subject to restrictions of no practical importance.

In a practical case such as that of the 204 measurements of resistance, we do not have the *a priori* distribution function, but if we are dealing with random samples, we can estimate the two desired characteristics of the distribution function for any given statistic. However, we do not usually know whether or not the data satisfy this condition. In fact, it is this very point that we want to settle. At first it looks like a hopeless tangle and it is, I believe, unless we invoke the element of judgement and the human faculty of postulation.

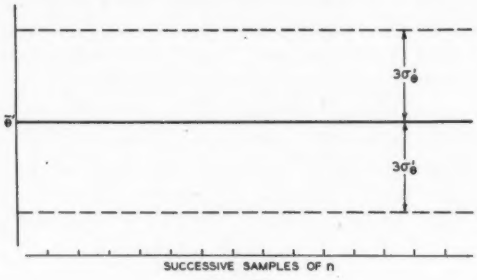

FIG. 13

Without more ado, we shall state two postulates, the reasonableness of which we cannot consider here.

Postulate 1: When we have done everything that we can do to find and to eliminate the chance or unknown causes of variation in a statistic θ of a sample of size n from sample to sample, the resultant variation in the statistic will be produced by a constant system of chance causes.

Postulate 2: It is humanly possible to find and eliminate causes of variability in the statistic until the remaining causes represent a constant system.

In what we shall say henceforth we shall assume that the constancy of a cause system applies to a single observation in the sample or, in other words, that we are dealing with the second class of random samples previously defined.

Suppose now that we have a set of N observed values that we wish to test to see whether or not it is likely that one observation arose under the same essential conditions as any other. It can be shown that it is reasonable to believe that if there is no *a priori* rational basis for subdividing these data into groups, then statistical theory is perhaps almost useless in helping us to decide the issue. If, however, there is a basis for dividing the total set of N observed values into m rational subgroups, the theory of random sampling does something very useful indeed. It tells us how to set up certain limits $\bar{\theta} \pm t\sigma_\theta$ for samples of size n from each of the m rational subgroups in such a way that, when an observed value θ falls outside these limits, it is unlikely that the individual observations arose under the same essential conditions.

Far more important, however, is the fact that this theory enables one to do economically what he wants to do by making possible the establishment of limits $\bar{\theta} \pm t\sigma_\theta$ within which observed values of a statistic θ for samples of size n

should fall with a probability P approximately equal to any previously desired probability P' *provided he has succeeded in eliminating causes of variability which can and should be found and weeded out.* Furthermore, these limits are such that

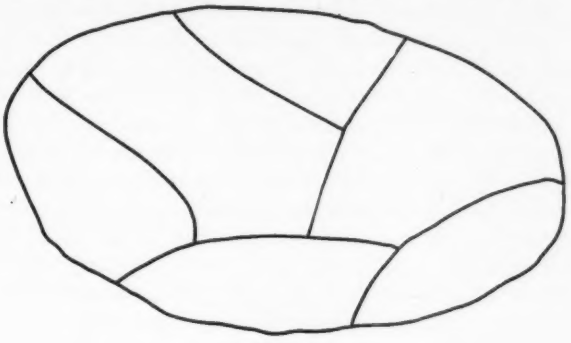

FIG. 14

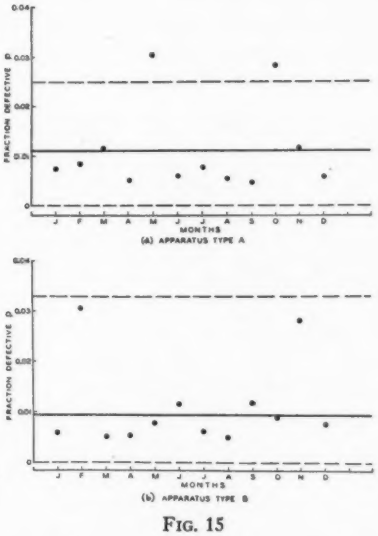

FIG. 15

if these causes do exist, it is likely that the probability $1-P$ of detecting them will be greater than $1-P'$ whereas in the long run, if the causes do not exist,

$$\lim_{n\to\infty} \bar{\theta} = \bar{\theta}' \quad \text{and} \quad \lim_{n\to\infty} \sigma_\theta = \sigma_\theta',$$

so that P approaches the desired value P'.

Hence, if we are to take a sample of n observations about which we are to ask later whether or not each observation has been taken under the same essen-

tial conditions, we should first of all divide the objective set of observations into rational subgroups. Schematically we may think of the boundary of Fig. 14 as representing the universe of possible measurements and the subdivisions as representing the boundaries of the rational subgroups. We should then divide the sample of size n into subsamples, taking one subsample from each of the m rational subgroups where, of course, it is assumed that $n \geqq m$. We cannot here go into the details of the method of doing this under practical conditions.

It must suffice to give illustrations of how this method has been found to work in practice. For example, applying this test to the variations in Fig. 9, we get the control charts of Fig. 15. Although the variations in the qualities of these two types of apparatus appear to be about the same, we see that the test indicates trouble in one case and not in the other. The indication of trouble was found to be justified.

Similarly, applying the test to the data of Table 2, we find, Fig. 16, that the variations in resistance are such that they should not be left to chance as indicated by points falling outside the limits in the left side of this figure. In this case it is interesting to note that further research found at least some of the assignable causes of variability and removed them with the result that the variability then fell within the control limits as indicated in the right side of Fig. 16.

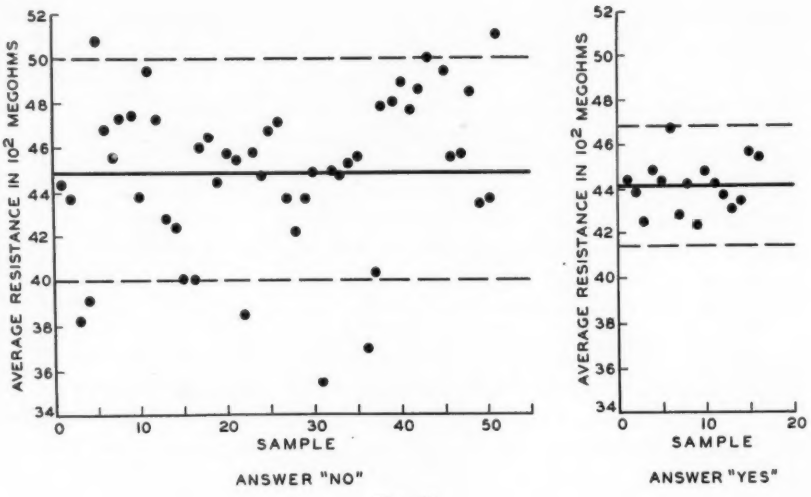

FIG. 16

If we apply this criterion to a set of data which *a priori* we have every reason to believe arose under a constant system of chance causes, such as Millikan's measurements of the charge on an electron, all of the points should lie within the dotted limits. They do, at least in this instance, Fig. 17.

4. *More About Applications.*

We may now consider further the contributions of the theory in answering some of the specific questions raised earlier in this paper. For example, how does the theory help the civil engineer to estimate the maximum run-off in terms of the data of Table 1 and the inspection engineer to establish plans for sampling the poles of Fig. 7 and the soldered terminals of Fig. 8? Let us consider the dam problem first.

A knowledge of the theory certainly makes an engineer very cautious about trying to infer anything about the maximum run-off through the use of the theory of estimation unless he feels sure that the causes of variation in run-off constitute a constant system of chance causes—and I fear that the causes do not usually behave in this way. On the other hand, the theory provides the engineer with a method for testing whether or not the data did come from a constant system of chance causes.

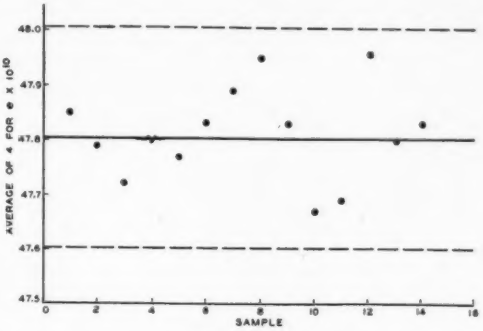

FIG. 17

Incidentally, it is of interest to note that, even assuming constant causes of variation in the run-off, such theory indicates that some of the attempts made in the literature to estimate maximum run-off through the use of high-brow frequency curves cannot hope to do much more than simply befog the issue by the welter of complicated formulas with their parameters almost as numerous as the available data themselves. On the positive side, however, the theory directs attention to the importance of estimating the average $\overline{X}'$ and standard deviation σ' of the universe corresponding to the assumed constant system of chance causes and of using these *intelligently* in the theorem of Tchebycheff.

In the inspection of poles and terminals, we wish to find out as much as possible about a finite set of N things by looking at only n of them. I think you will agree that in neither case is it feasible to tag all of the poles or all of the terminals shown in Figs. 7 and 8 and to set up a bowl experiment whereby we may get a random sample. What we must do under such conditions is to try to divide on *a priori* grounds the objective set of N things into rational subgroups. Then we must take some from each of the m rational subgroups so that the

sample of size n to be taken will be made up of component parts $n_1, n_2, \cdots, n_i, \cdots, n_m$. We can then use the theory of estimation for going from each subsample to its corresponding universe. By putting these predictions together we can go from the sample of size n to the lot N.

Thus, the theory shows us how to take a sample. It also tells us how many to take. More important, however, is the fact that it enables us to do what we want to do *economically*. For example, it helps a manufacturer to attain the following five economic advantages:[1]

1. Reduction in the cost of inspection.
2. Reduction in the cost of rejections.
3. Attainment of maximum benefits from quantity production.
4. Attainment of uniform quality even though inspection test is destructive.
5. Reduction in tolerance limits where quality measurement is indirect.

5. *Interplay of the Five Components in the Theory.*

The formal theory of specification, distribution, and estimation starts with numbers supposed to represent magnitudes of physical characteristics or entities. The operation of the formal part is independent of the nature of these entities and naturally cannot tell us whether or not we have established a set of numbers which represent the real entities in which we are interested. The choice of such a set of numbers is a function of the human mind.

To illustrate—what we sense through any one of our senses depends partly upon our previous use of these senses. For example, a child looking at a straight stick extending beneath the surface of a pool of water sees a bent stick. Similarly, the first time you saw what is shown in Fig. 18, you saw the length of the

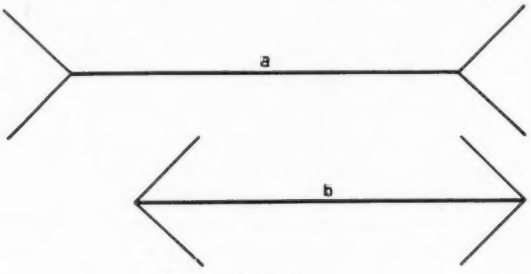

FIG. 18

line a to be different from the length of the line b. Without experience we see as one without experience.[2]

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[1] For further discussion of this subject see "The Economic Quality Control of Manufactured Product," by W. A. Shewhart, Bell Telephone Laboratories Monograph 496, June, 1930.

[2] As William James so aptly puts it: "And the random irradiations and resettlements of our ideas which supervene upon our experience and constitute our free mental play are due entirely to the secondary internal processes which vary enormously from brain to brain even though brains be exposed to the same 'outer relations'." *Principles of Psychology*, Vol. II, p. 638.—Whitehead puts the same idea in slightly different words: "One main law which underlies modern progress is that,

Hence in the establishment of a set of numbers we must make sure that the mind exerts its proper influence. Whether or not we start with the proper set of numbers or, in other words, good data, in this sense depends almost exclusively upon the human element. It does not appear to depend in any way upon the formal mathematics or logic involved in the theory of specification, distribution, and estimation.

Not only in choosing the method of expressing our experience quantitatively in the form of numbers does judgment enter. We have seen that it must play an important rôle in dividing the objective set of conditions into rational subgroups. Here again the mind of man, his imagination, his intuition, his store of facts, his prehensions, and his knowledge of the world in which he lives function almost exclusively.

It is, however, not only in the taking of the data that human judgment functions. When we come to set up control charts for a given statistic, we must make further choices. Three important ones are:

1. The choice of statistics to be used.
2. The choice of the probability P' associated with the limits for a given statistic.
3. The choice of the way of using the statistics.

In making such choices, the mind alone is *not nearly so powerful* as mind plus the formal logical and mathematical theory of specification, distribution, and estimation.

I can do no more than briefly illustrate what I mean. In choosing a statistic from the indefinitely large number that might be chosen, we must consider the usefulness of such a statistic together with the cost of computing it. One of the factors which enters into such a choice is the variability of the statistic measured in terms of the variability of some statistic taken as standard. Such a measure has been called efficiency.

In this connection the comparatively recent theory of distribution becomes of great value in that it makes possible the comparison of the efficiencies of various estimates of a given statistic and in the end the choice of that one which it is most economical to use. For example, Fig. 19 shows the approximate efficiencies of the median and $\frac{1}{2}$(max.+min.) relative to the efficiency of the arithmetic mean taken as unity over the range of sample size from two to infinity.

Similarly, Fig. 20 shows the variation in efficiencies of the mean deviation and the range as estimates of the standard deviation in respect to the variation

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except for the rarest accidents of chance, thought precedes observation. It may not decide the details, but it suggests the type. Nobody would count, whose mind was vacant of the idea of number. Nobody directs attention when there is nothing that he expects to see. The novel observation which comes by chance is a rare accident, and is usually wasted. For if there be no scheme to fit it into, its significance is lost. The way of thoughtless nature is by waste—a million seeds, and one tree; a million eggs, and one fish. In the same way, from a million observations of fact beyond the routine of human life it rarely happens that one useful development issues." *The Function of Reason*, p. 57.

of the root mean square deviation taken as unity.[1] Sometimes, of course, the economics of the situation, as for example the cost of computation, may counterbalance the efficiency of a statistic, particularly when the sample sizes are small. Hence it is essential that curves such as those presented in Figs. 19 and 20 should

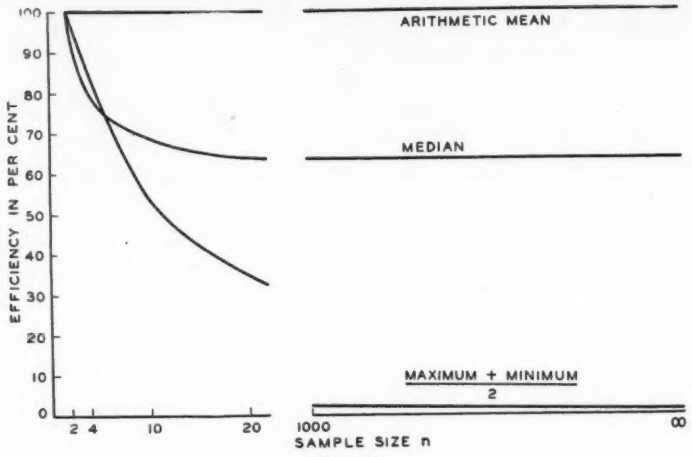


FIG. 19

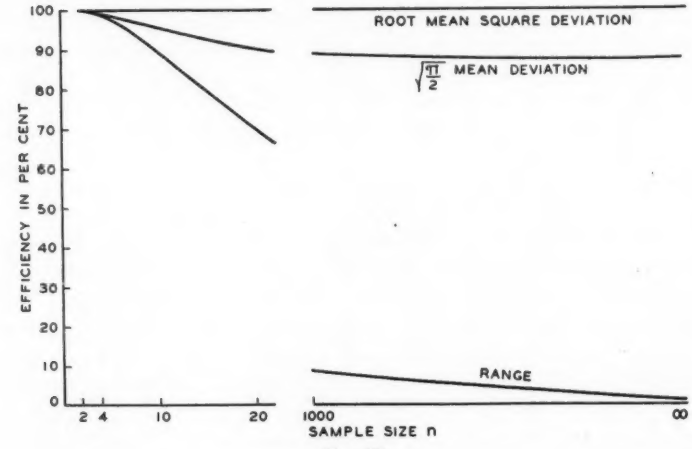


FIG. 20

be available to the applied statistician, if he is to make economical application of the theory. We see that, for comparatively small sample sizes (which for reasons we cannot go into here must often be used in control work because they

[1] Without the developments in the theory of distribution of the last decade these curves could not have been drawn.

catch trouble when larger samples would not), the efficiencies of the various estimates are not so widely different.

Moreover, recent theory enables us to compare the advantages of different ways of using statistics. For example, if we are to make use of the average and standard deviation, we may construct control charts on each statistic separately or we may construct an enclosed area in a plane based upon a consideration of the probability of the simultaneous occurrence of a given average and a given standard deviation. Or again, we may construct control charts upon the basis of differences in averages and differences in standard deviations. As an example, Fig. 21 illustrates the first two methods. On the basis of the first method, if an observed point such as either of the two shown in this figure, falls outside the

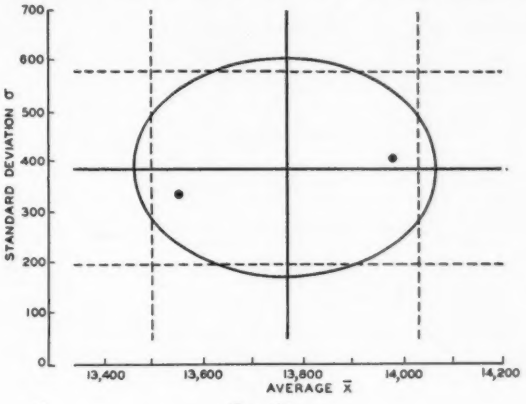


FIG. 21

dotted limits, this is taken as an indication of lack of constancy of the cause system. Similarly, if any point falls outside the ellipse, it is taken as a similar indication based upon the second method of using the statistics. It appears that there is always one of these many ways which is considerably better from a practical viewpoint than the others and it is the use of modern theories of distribution that enables one to choose the best way of using the statistics.

V. CONCLUSION

We started out to consider two questions often asked by the practical man about the theory of random sampling—"What is it?" and "What's it good for?" We have tried to throw a little light on the kind of answers that can be given and that seem to be of interest to the scientist and industrial artisan faced with practical problems that are not usually treated in books touching upon the theory of sampling.

Summary answers proposed are:

A. The theory of random sampling is that part of the scientific method which enables one to predict the future in terms of the past when the future is

related to the past through the Law of Large Numbers. The theory consists of a statement of the more or less accepted methods of using the human mind, and certain formal logical and mathematical elements classified usually under the subjects of specification, distribution, and estimation.

B. The useful purpose served by the theory is to define clearly those conditions under which prediction of the future can be made from a sample and to set forth definite procedures for predicting when possible.

Emphasis has been laid on the application of the theory in helping us to do what we want to do by telling us when variability need not be left to chance and in helping us to do this economically. Perhaps a little light has been thrown on the answer to another question: Why does the theory not enjoy broader application? Possibly there is need for a little shuffling of emphasis in discussions of the theory in the literature. Be that as it may, industry owes a debt of gratitude to the mathematicians who are doing so much to extend our knowledge of the subjects of specification, distribution, and estimation.

A NOTE ON THE PRINCIPLES OF MECHANICS[1]

By H. M. DADOURIAN, Trinity College

When the foundations of mechanics are cleared of special laws and theorems there remains a single fundamental principle from which the entire science of mechanics may be developed. The principles of *virtual work*, of *least action* and of *least constraint* are different forms of this principle. In some books on advanced mechanics this fact is recognized and made use of with great success in developing the subject in an orderly and logical manner, somewhat similar to the development of geometry from postulates. Lagrange's Mecanique Analytique and Hertz's Principles of Mechanics are conspicuous examples of such books.

Authors of elementary mechanics have ignored this important fact and have produced text-books which consist largely of collections of special theorems, laws and methods for solving problems. These authors usually give Newton's three laws of motion but they make little or no direct reference to them in the course of the presentation of the subject.

It is not to be wondered that these authors do not attempt to use Newton's three laws as the postulates of mechanics. Critics, such as Mach, Karl Pearson and Hertz, have severely criticised Newton's laws and have claimed that they are not satisfactory. They have pointed out that the first law is only a special case of the second, that the second law is nothing more than a definition of force, that the third law, while being a true principle, is not sufficiently broad to be used as a foundation of mechanics.

The fact that for sixty years after the publication of Newton's *Principia*

[1] This note will appear as an appendix in the third edition of the author's *Analytical Mechanics*.

students of mechanics had to appeal to all sorts of special theorems in order to solve problems of rigid bodies indicates that there is some basis to these criticisms. In commenting on D'Alembert's principle Routh (Elementary Rigid Dynamics, p. 47) says, ". . . It required the exercise of ingenuity and skill to detect the most suitable (theorem) in each case. Such problems were for some time a sort of trial of strength among mathematicians. The *Traité de Dynamique* published by D'Alembert in 1743 put an end to this kind of challenge by supplying a direct and general method of solving, or at least throwing into equations, any imaginable problem. The mechanical difficulties were in this way reduced to difficulties of pure mathematics."

These criticisms have produced little or no effect on the presentation of elementary mechanics, for two reasons. The first reason is obviously inertia and traditionalism. The second and more important reason is the fact that the well known forms of the principle are too difficult for the beginner to understand and demand too much mathematical knowledge. D'Alembert's principle is probably the easiest, but the six differential equations, which form this principle, are too forbidding to be handed to the beginner.

Before proceeding further I must confess that I have hesitated a very long time before deciding to write this note, because what follows may be construed as self-advertisement. But the subject seemed to me important enough to justify running this risk.

In the second edition of my Analytical Mechanics I expressed the underlying principle of mechanics in a form which is not forbidding to the beginner. It is broad enough to form a sound basis from which the entire subject may be developed in logical manner. This form of the principle, which I have called the *action principle*, states:

The vector sum of all the external actions to which a system of particles or any part of it is subject at any instant, vanishes.

$$\sum A = 0.$$

Two types of actions are postulated, namely *forces* and *kinetic reactions*. A force is defined as the action of one particle upon another particle. A kinetic reaction is defined as the action of the material universe as a whole upon a particle as a result of the acceleration of the latter. It is equal to the product of the mass of the particle and its acceleration, and has a direction opposite to the acceleration. This distinction between forces and kinetic reactions removes once for all the confusion which has prevailed in connection with uniform circular motion, as indicated by the term *centrifugal force*.[1] It makes very clear the fact that there is a complete agreement between the dynamic equilibrium of a particle in linear motion with acceleration and the dynamic equilibrium of a particle in uniform circular motion, and that in both cases the particle is in

[1] Hertz has a very interesting discussion of this subject in his *Principles of Mechanics*, p. 5.

dynamic equilibrium under the action of the resultant force and an equal and opposite kinetic reaction.

Introducing forces and kinetic reactions explicitly in the last equation, we obtain

$$\sum(\boldsymbol{F} - m\dot{\boldsymbol{v}}) = 0. \tag{A'}$$

This vector equation may be split into the following three scalar equations.

$$\sum(X - m\ddot{x}) = 0, \quad \sum(Y - m\ddot{y}) = 0, \quad \sum(Z - m\ddot{z}) = 0.$$

The action principle is equivalent, therefore, to the first three equations of D'Alembert's principle. It has, however, a number of advantages over D'Alembert's principle. The action principle is stated in spoken language. It states explicitly that we are dealing with *vector* magnitudes; that we need consider only *external* actions and consequently may neglect internal forces; that we need consider only those actions to which the body, whose equilibrium or motion is being studied, is *subject*, and consequently may neglect the forces which that body exerts upon other bodies; and finally that the actions at different *instants* must be considered separately.

The pedagogical value of these advantages can hardly be exaggerated. The student, in his attempt to solve a problem in equilibrium, often represents not only the forces which are acting upon the body in equilibrium but also those which it exerts upon bodies it is in contact with. As a result he gets into difficulties. The action principle, which states explicitly that only those forces are to be considered *to which the body is subject*, helps avoid such difficulties when this principle is made the starting point of the problem.

Another advantage of the action principle lies in the fact that it emphasizes actions and not forces. The beginner is familiar with action in the form of push, pull and resistance, but he has only a vague conception of the idea of force. The concept of force is an abstraction which is difficult for the beginner to grasp fully. Consequently when he thinks in terms of forces he does strange things. He introduces forces where none existed and leaves out of consideration forces which actually exist. Ask a student to represent the forces acting upon a bullet in the projectile problem, for example, and you will find that he is just as likely as not to place a force back of the bullet pushing it forward in its path. If he were thinking in terms of action he could not make such an absurd blunder because his experience and intelligence would tell him that no body follows the bullet in order to push it forward. Again, in a problem, say, on the equilibrium of a body on an inclined plane he is likely to forget that the plane exerts a force upon the body. He is not likely to make such a blunder if he thinks in terms of actions and asks himself, "What bodies are acting upon the body under consideration?"

The action principle lends itself readily to a gradual development of the subject. It is easily understood by the beginner, at least to the extent of his needs at the different stages of his development as a student of mechanics. While

studying the equilibrium of a particle, for example, he need only be reminded that $\dot{v}=0$ in order to be able to obtain $\Sigma F=0$ as the condition of equilibrium. Newton's second law of motion, $F=m(dv/dt)$, is obtained by assuming that the system is a single particle acted upon by the resultant force F. Newton's first law is obtained by letting $F=0$. Other important laws and theorems, such as Newton's third law of motion, the principle of the conservation of dynamical energy, the theorem on the motion of the center of mass of a system of particles and the principle of the conservation of linear momentum are progressively derived from the action principle.

In order to put the action principle in a form adapted to motion of rotation, a new concept is introduced, namely, that of *angular action*. An angular action is defined as the moment of a linear action relative to a given exis. The angular action of a force is called a *torque*. The angular action of a (linear) kinetic reaction is called an *angular kinetic reaction*. Then the following equation is derived from equation (A).

$$\sum(\boldsymbol{G} - m\boldsymbol{p}\dot{\boldsymbol{v}}_p) = 0, \tag{Aa}$$

where $\boldsymbol{G}$ represents the moments of external forces, $\dot{v}_p$ the projection of accelerations upon a plane perpendicular to the axis of reference, $-m\boldsymbol{p}\dot{\boldsymbol{v}}_p$ the angular kinetic reactions of the particles of the system, and $\boldsymbol{p}$ the lever arm of $-m\dot{\boldsymbol{v}}_p$. Equation (Aa) states:

The vector sum, relative to any axis, of all the external angular actions to which a system of particles or any part of it is subject at any instant vanishes.

$$\sum \boldsymbol{A}_a = 0.$$

This new principle is called the *angular action principle.*

When the vector equation (Aa) is resolved into three scalar equations the following are obtained.

$$\sum m(yZ - zY) = \frac{d}{dt}\sum m(y\dot{z} - z\dot{y}),$$

$$\sum m(zX - xZ) = \frac{d}{dt}\sum m(z\dot{x} - x\dot{z}),$$

$$\sum m(xY - yX) = \frac{d}{dt}\sum m(x\dot{y} - y\dot{x}).$$

The angular action principle is thus equivalent to the last three equations of D'Alembert's principle. The angular action principle, however, has a number of advantages over these equations. As in the linear action principle the important concepts represented by the terms *vector, external, subject to,* and *at any instant* are made explicit. The conditions of equilibrium of a rigid body, the

equation of motion of rotation of a rigid body, the theorem on the motion of a system relative to its center of mass, the principle of the conservation of angular momentum are easily derived from equation (Aa).

The main advantage of equation (Aa) over the corresponding equations of D'Alembert, so far as elementary mechanics is concerned, is derived from the fact that the only type of rotation which is studied in elementary mechanics is uniplanar rotation. For, in uniplanar motion $\dot{v}_p = \dot{v}$ and equation (Aa) reduces to the simpler form

$$\sum(G - mp\dot{v}) = 0,$$

which may be considered either as a vector or a scalar equation, and consequently does not have to be resolved into component equations. In the case of a rigid body the last equation is readily changed to

$$G = \frac{d}{dt}(I\omega),$$

where G denotes the resultant torque due to the external forces acting on the body.

The two action principles are equivalent, that is, each may be derived from the other. One is especially adapted to the mechanics of translation and the other to the mechanics of rotation. Each is so phrased as to emphasize the perfect analogy which exists between motion of translation and motion of rotation, and between the corresponding dynamical magnitudes in these two types of motion.

In equation (A) mass is supposed to be constant. If it is taken to be variable this equation is replaced by the more general equation

$$\sum\left[F - \frac{d}{dt}(mv)\right] = 0.$$

This does not involve, however, a change in the wording of either of the action principles, because when mass is variable the kinetic reaction equals $-(d/dt)(mv)$. In this case the angular kinetic reaction becomes $-p(d/dt)(mv_p)$, and equation (Aa) takes the form

$$\sum\left[G - p\frac{d}{dt}(mv_p)\right] = 0.$$

NUMERICAL SOLUTION OF LINEAR EQUATIONS BY VECTORS

By J. P. BALLANTINE, University of Washington

In two previous articles in this Monthly[1] I have developed certain well known theoretical facts about linear equations by the use of vectors. In the present note a new and practical method of obtaining numerical solutions of systems of linear equations is found to have as its basis the same notion of vectors.

The vector method of solution involves a large number of very simple steps. At the end of each step it is possible to read off an approximate solution together with a measure of precision. Thus the work can be carried to any desired degree of accuracy. At the end of each step a complete check is afforded against any possible mistakes of computation. When the coefficients are integers and the method of determinants yields exact rational values for the unknowns, the vector method also yields the exact solutions.

For the purpose of illustration, take the example:

$$\begin{aligned} 12x_1 - 3x_2 + 10x_3 + 6x_4 &= 12, \\ 21x_1 + 16x_2 - 8x_3 + 18x_4 &= 23, \\ 7x_1 + 18x_2 + 12x_3 - 5x_4 &= -15, \\ 18x_1 + 4x_2 + 17x_3 - 9x_4 &= 12. \end{aligned} \tag{1}$$

These four equations may be replaced by the single vector equation:

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = A_5, \tag{2}$$

where

$$\begin{aligned} A_1 &= (12, 21, 7, 18), \\ A_2 &= (-3, 16, 18, 4), \\ A_3 &= (10, -8, 12, 17), \\ A_4 &= (6, 18, -5, -9), \\ A_5 &= (12, 23, -15, 12). \end{aligned} \tag{3}$$

Numerical values of x_1, x_2, x_3, x_4, may be obtained by first solving the homogeneous equation:

$$A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4 + A_5y_5 = 0, \tag{4}$$

and dividing the value of y_1, y_2, y_3, y_4, by $-y_5$.

An approximate solution of (4) can be obtained by finding a linear combination of the vectors A_1, A_2, A_3, A_4, A_5 which is at most a short vector. In the present case $-A_1+A_5$, which we shall call A_6, is a short vector, at least it is shorter than A_5. Hence A_5 is replaced by A_6, and from now on we work with the

[1] *A graphical derivation of Cramer's rule*, in vol. 36 (1929), pp. 439–441, *The theory of least squares by vectors*, in vol. 37 (1930), pp. 25–26.

set A_1, A_2, A_3, A_4, A_6. In a similar manner, $A_2+A_6=A_7$ is used to replace A_2. It is obvious that each new vector obtained is a linear combination of the original set, and as the vectors get shorter we obtain better approximate solutions of (4).

The information relating to each vector is given in one row of the following computation. First we give the number of the vector. Next the numbers of the two vectors added or subtracted to yield that vector. Next we give the coefficients of that vector when stated as a linear combination of the original set. Next the components of the vector. Next the norm, or square of the length, of the vector. It is useful to have the norms, for at every step one must decide which of two vectors is shorter. The norms, however, need not be computed exactly. In the final column we give the number of the vector which finally replaces the vector of that row. At every step we have five vectors, and each new one formed replaces a longer one obtained earlier.

		A_1	A_2	A_3	A_4	A_5					
1		1					12	21	7	18	958	9
2			1				−3	16	18	4	605	7
3				1			10	−8	12	17	597	12
4					1		6	18	−5	−9	466	8
5						1	12	23	−15	12	1042	6
6	5−1	−1				1	0	2	−22	−6	524	14
7	6+2	−1	1			1	−3	18	−4	−2	353	11
8	7−4	−1	1		−1	1	−9	0	1	7	131	
9	7−1	−2	1		0	1	−15	−3	−11	−20	755	10
10	9+3	−2	1	1	0	1	−5	−11	1	−3	156	20
11	10+7	−3	2	1	0	2	−8	7	−3	−5	147	22
12	11+3	−3	2	2	0	2	2	−1	9	12	230	13
13	12+11	−6	4	3	0	4	−6	6	6	7	157	15
14	13+6	−7	4	3	0	5	−6	8	−16	1	357	16
15	13−8	−5	3	3	1	3	3	6	5	0	70	
16	15+14	−12	7	6	1	8	−3	14	−11	1	327	17
17	16+10	−14	8	7	1	9	−8	3	−10	−2	177	18
18	17−11	−11	6	6	1	7	0	−4	−7	3	74	19
19	18+15	−16	9	9	2	10	3	2	−2	3	26	
20	19+10	−18	10	10	2	11	−2	−9	−1	0	86	21
21	20+15	−23	13	13	3	14	1	−3	4	0	26	
22	21+11	−26	15	14	3	16	−7	4	1	−5	91	23
23	22+19	−42	24	23	5	26	−4	6	−1	−2	57	24
24	23+21	−65	37	36	8	40	−3	3	3	−2	31	

Thus, reading opposite 11 in the first column, we find the notation 10+7. This means that the vector A_{11} was computed by the formula $A_{11}=A_{10}+A_7$.

The numbers $-3, 2, 1, 0, 2$ indicate that A_{11} can be expressed as a linear combination of the original set of vectors by the formula

$$-3A_1 + 2A_2 + A_3 + 0A_4 + 2A_5 = A_{11}. \tag{5}$$

These coefficients were computed by adding the corresponding coefficients for the vectors A_7 and A_{10}. The numbers $-8, 7, -3, -5$, appearing next in the same row of the computation are the components of the vector A_{11}, and are computed by adding the components of the vectors numbered 10 and 7. They may also be checked by (5). The sum of the squares of the components of A_{11} is next shown to be 147. Of the two vectors A_7 and A_{10} used in forming A_{11}, A_7 is the longer, having a norm of 353, and is replaced by A_{11}. This fact is indicated by entering 11 in the last column opposite 7 in the first column. In a similar manner, A_{11} itself is later replaced by A_{22}.

The computation has been carried to a point where all the vectors have been replaced except A_8, A_{15}, A_{19}, A_{21}, A_{24}. The computation could be continued by forming $A_{24}-A_{15}$. However, this is not necessary as a good solution can now be read off by inspection. In equation (5), neglecting A_{11}, we see that $-3, 2, 1, 0, 2$ is an approximate solution of equation (4). It fails to satisfy equation (4) by the vector A_{11}. Dividing the first four values of this solution by the negative of the last one, i.e., by -2, we have $1.5, -1, -.5, 0$ as an approximate solution of (2). It fails to satisfy (2) by the vector $-A_{11}/2$, whose norm is $147/4$. A better solution is obtained by reading opposite 19. A_{19} is expressed as a linear combination of the original set by the coefficients $-16, 9, 9, 2, 10$, and since A_{19} is short, these coefficients are an approximate solution of (4). Dividing by the negative of the last of them, we obtain $1.6, -.9, -.9, -.2$ as an approximate solution of (2). It fails to satisfy (2) by $A_{19}/10$ which has a norm $26/100$.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE DIRICHLET FORMULA, AND INTEGRATION BY PARTS

By L. M. GRAVES, University of Chicago

It seems not to have been generally noticed that the calculus formula for integration by parts is a special case of the Dirichlet formula for interchange of order of integration in an iterated integral, namely,

$$\int_a^b \int_a^x F(x, s)ds\,dx = \int_a^b \int_s^b F(x, s)dx\,ds.$$

This relation holds true for either Riemann or Lebesgue integrals. The essential idea can be indicated by specializing the indefinite integrals as follows. Let

$$f(x) = \int_a^x f'(s)ds, \quad \int_b^x g(s)ds = h(x).$$

Then

$$\int_a^b f(x)g(x)dx = \int_a^b \int_a^x f'(s)g(x)ds\,dx$$

$$= \int_a^b \int_s^b f'(s)g(x)dx\,ds = -\int_a^b f'(s)h(s)ds.$$

Professor C. J. Coe has used the proof of this relation as an exercise for some of his classes at the University of Michigan. The relation first came to my attention through a calculus of variations problem in which the first variation could be transformed by the use of Dirichlet's formula, but not by the usual integration by parts.

RECENT PUBLICATIONS

EDITED by ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematische Quellenbücher. IV, Infinitesimalrechnung. Heinrich Wieleitner. Berlin, Otto Salle, 1929. 160 pages. Price, 4.50 RM.

This little and inexpensive work by Dr. Wieleitner is the fourth mathematical source book in a series of handbooks edited by Drs. Elwald Wasserloos and Georg Wolff. The purpose of each of these works is to give a small number of the most important sources relating to some particular branch of knowledge. In this volume (No. 24 of the large series) upwards of twenty of the most interesting source items relating to the differential and integral calculus are given. These include such topics as the following: (1) The Archimedean axiom, with source quotations from Archimedes and Euclid. Substantially, Archimedes says (*De Sphaera et Cylindro*, I, 5) that of unequal magnitudes the greater exceeds the less by such a magnitude as, when continually added to itself, can be made to exceed any assigned magnitude of the same kind, and Euclid, V, Def. 4, involves the same assumption. (2) The ratio of the areas of two circles, from Euclid XII, 2. (3) The quadrature of the parabola, from Archimedes. (4) The refutation of the idea of indivisible parts of a line (*von den Atomlinien, De insecabilibus lineis, Atomos grammos*) by Aristotle. (5) The sum of square num-

bers, by Archimedes. (6) The volume of the spheroid, by Archimedes. (7) The volume of the sphere, by Archimedes. (8) The volume of the sphere in the 17th century (Luca Valerius). (9) The "apple-shaped body" of Kepler—the second great step, of which Guldin's was the third, towards the barycentric calculus. (10) A selection from Cavalieri's *Geometria Indivisibilibus* (1635; 2d. ed., 1653). (11) A selection from Torricelli's "De solido hyperbolico acuto," in the *Opera* of 1644. (12) The quadrature of hyperbolas in general, from Fermat. (13) Fermat's method of maxima and minima. (14) Pascal's "characteristic triangle" and its application to the problem of the quadrature of the circle and trigonometric integrals. (15) Newton's step towards considering differentiation and integration as inverse operations. (16) Leibniz's first printed rules for differentiation. (17) Newton's application of infinite series to the representation of the arc-sine, sine, and cosine. (18) Leibniz on the "arithmetic quadrature" of the circle. (19) Jean (I) Bernoulli's problem of tangents to the Archimedian spiral. (20) Newton's discovery of fluxions. (21) Euler's differential quotient of the sine.

It will be observed that Dr. Wieleitner has here indicated the most important steps taken in the development of the calculus up to about the close of the 18th century when, for practical purposes the theory was complete, although more remained in laying firm the foundations. He has not only given translations into German of important passages of the makers of the discipline, but he has added a commentary on each, explaining by the aid of modern symbols its significance. This has been done in a condensed form that will serve to give to students of the subject a good knowledge of the development of the method upon which rests so much of the advanced mathematics of the present day.

DAVID EUGENE SMITH

Elementary Theory of Finite Groups. By L. C. Mathewson. Houghton Mifflin Co., 1930. x+165 pages.

This text is planned for the beginning student of finite groups. The material is arranged for teaching purposes. Each unit of work is followed by a set of exercises and at the end of each chapter there are additional exercises of a more advanced nature which supplement the theorems of the text. The whole book is copiously illustrated with examples and easily read.

Perhaps a summary by chapters of the subjects discussed will indicate what field the text covers. The reader is introduced very gradually to the idea of a group. In fact the first chapter is devoted to simple examples of finite groups and to the set of postulates defining a group. The next two chapters take up permutations and permutation groups, defining such terms as transitivity and primitivity. A chapter each is devoted to abelian groups, to groups of isomorphisms, and to some of the elementary theorems of abstract group theory, whereas two further chapters are devoted to some more examples and definitions of abstract groups and certain special groups. The book closes with two chap-

ters on groups of linear substitutions and some applications of group theory, such as the Galois theory of equations, constructions by ruler and compass, and the Lie theory of one-parameter groups.

These last two chapters are very brief, but this is the intention of the author. The chapter on the applications is chiefly an outline, while the author states that the chapter on groups of linear substitutions "aims only to mention a few things in the Frobenius theory of groups of linear substitutions." No mention, except in a problem of a set of exercises, is made of the Frobenius group characters or characteristics. Consequently, some interesting theorems due to this theory are omitted. It would seem more valuable perhaps to the student to omit the chapters on the applications, which for lack of space are necessarily very sketchy, and to substitute a more detailed account of the theory of groups of linear substitutions and group characteristics. An omission of one of the methods of group theory seems unfortunate.

Some theorems, such as the one on page 44 on a k-ply transitive abelian group, seem somewhat trivial. In this theorem k is obviously restricted to one, for a permutation which fixes a letter a and displaces a letter b is not commutative with a permutation which replaces the letter a by the letter b. One of the elementary and recent theorems, namely P. Hall's extension of Sylow's theorem for solvable groups (Journal of the London Mathematical Society, 1928) might profitably be mentioned. However the elementary character and the brevity of the book cause inevitable limitations. On the whole the book is carefully written and well arranged.

MARIE J. WEISS

Infinite Series. By Tomlinson Fort. Oxford University Press, 1930. iv+253 pp.

There is a great need for an elementary text on infinite series since the lack of manipulative skill and of mathematical maturity on the part of most of the students make the books of Bromwich and of Knopp less suitable for introductory courses. Professor Fort's book fills this gap, to some extent at least.

The book covers extensive ground. The following extract from the table of contents shows what subjects are treated, the number of pages being given in parentheses after the title.

I. The number system (6); II. Sequences (11); III. Convergent and divergent series (7); IV. Series whose terms are positive (24); V. Series some of whose terms are positive and some negative (5); VI. Series whose terms are complex (7); VII. Transformations of series and operations with series (18); VIII. Multiple series (13); IX. Uniform convergence (27); X. Continuity and integrability. Quasi-, sub-, and infra-uniform convergence (8); XI. Power series (27); XII. Dirichlet series (21); XIII. Binomial coefficient series (2); XIV. Factorial series (4); XV. Generalized factorial series (6); XVI. Fourier series (15); XVII. Summation of divergent series (45); XVIII. Asymptotic series (7).

There are obviously some deviations here from the conventional plan, and

one or two of the inclusions are of doubtful value, but on the whole the book is well planned. It does not demand advanced knowledge or mastery of technique from the student. There is an ample supply of exercises and few if any of the tricky Tripos type. The author states clearly what are the hypotheses and what are the conclusions in every theorem, a helpful feature for the student, but slightly tedious when repeated 244 times. The book contains some interesting portions based on the author's own investigations, especially Chapter XV. This and Chapter XII add much to the value of the book.

A defect of the book is a certain indefinite monotony, and a lack of outlook and of historical perspective. The references to the literature are also far too scanty—less than a dozen books or papers on infinite series are mentioned from cover to cover. The author at times shows a surprising vagueness due no doubt to his endeavor to avoid certain advanced theories, such as measure, integration, even analytic functions. Much is said about differentiation and integration of functions of a complex variable throughout the book, but there is nowhere to be found a clear-cut definition of these operations, nor is the distinction between functions of a complex variable and analytic functions ever called to the reader's attention. One can understand the author wanting to avoid the Lebesgue theory in an elementary text, but he should then also avoid the subtleties of the Riemann integral and of continuity which are apt to be more difficult and less important than the Lebesgue theory. A couple of examples taken from Chapters IX and X will illustrate these points.

Thus theorem 124 on p. 111 deals with term by term differentiation of a series whose terms are functions of a complex variable. In the proof it is tacitly assumed that $u_n'(z)$ has a Riemann integral and is the derivative of its indefinite integral. This should be justified, or, if necessary, included among the hypotheses. Here, as in several other places, it is hard to tell if the author is exclusively concerned with analytic functions, or if he has also in mind general functions of a complex variable, or, in particular, functions of a real variable, the derivatives in question being taken along a curve or along the real axis as the case may be.

In theorem 133 on p. 124 the author wants to prove Arzelà's necessary and sufficient condition for term by term (Riemann) integration of a series of arbitrary functions. In trying to avoid sets of measure zero in the sense of Lebesgue, he introduces "discrete sets" instead, which are simply sets of Jordan content zero. His statements that the set of discontinuities of a Riemann integrable function is discrete, and that the sum of an enumerable set of discrete sets is also discrete, are obviously incorrect. The reviewer believes that it would have been much better to omit Chapter X altogether. The notions involved are rather complicated, and the author does not always treat them successfully. Even the terminology is at least debatable.

Various other points could be raised. Thus, the discussion of Fourier series is rather meager. On the other hand, there is no justification at the present time for devoting 11 pages out of 45 on the theory of summability, to Borel's integral

definition. It is an interesting method of great historical importance, but of comparatively little power. It would have been much better to devote some of this space to Cesàro's method for non-integral orders, and to give the rest to a study of analytic continuation by a method of summation which is really effective when applied to power series, e.g., that of LeRoy.

The printing is of the usual high quality which one associates with the Clarendon Press. The reviewer has found some misprints, but none of importance.

EINAR HILLE

College Algebra. By B. H. Crenshaw and D. C. Harkin. P. Blakiston's Son and Company, Philadelphia, 1929. xiii+224 pages.

The following are the chapter headings of this book in the order of their appearance: Review of fundamental concepts; Linear equations in two and three unknowns; Exponents; Quadratic Equations; Ratio and Proportion—Variation; Series, Binomial Theorem, and Mathematical Induction; Logarithms; Undetermined Coefficients and Partial Fractions; Numerical Equations; Theory of Equations; Theory of Numbers.

The first chapter is devoted to a review of fundamental concepts which according to the preface are "too often assumed to be well-understood." It is to be feared, however, that the first chapter may fail to relieve this condition.

In the preface the authors say: "In line with the dictum of Ernst Mach, that science tells not why but how, a large number of illustrative examples are offered, fully worked out and checked." Upon a first reading of the preface the reviewer interpreted this to mean that the theory was well illustrated by examples but a reading of the text convinced him at once that the authors meant quite literally what they said. Thus for the most part no logical development of a topic is attempted, the authors prefering to proceed at once to illustrative examples and then to direct the student to go and do likewise. The places where this policy has been deviated from convinces the reviewer that perhaps after all this was wise although in general he does not entirely agree with the authors' interpretation and application of Mr. Mach's dictum. Following the lead of the authors let us proceed with some illustrative examples.

After carrying out the ordinary elimination of y from two linear equations containing x and y, the authors proceed to relieve "determinants of the appearance of black magic" as follows: "By a detached coefficient method, we may, . . . , put the coefficients in a determinantal array and perform the same operations on its elements:

$$\begin{vmatrix} b & ax - e \\ d & cx - f \end{vmatrix} = 0.$$

The corresponding operations are then performed until zero is obtained in the upper left hand corner. Thus the term "determinant" appears for the first time

as an adjective and the actual concept introduced is the matrix of coefficients which, however, is set equal to a number. Without further explanation the "determinantal array" becomes on the next page a "determinant" and the expansion in terms of the elements of a row is (incorrectly) given. No explanation of the rule is attempted nor is one possible since no determinant has yet been defined.

Again in the chapter on series the terms series, sum of the series, and limit of the sum of the series are used in place of the almost universally adopted terms sequence, series, sum of the series, respectively. The comparison test is incorrectly stated inasmuch as the terms are not restricted to be positive. (Absolute and conditional convergence are defined for alternating series only.)

In the chapter on exponents the student is lead to believe that the meanings for non-integral exponents are derived from analogues in the operations of coefficients. In the discussion of irrational numbers are found the following two statements on the same page: "irrational number means an n-th root of a positive rational number which is not a perfect n-th power"; "rational numbers might be considered special cases of irrationals."

Illustrations and exercises on the four fundamental operations with complex numbers appear in the chapter on exponents. Thus, except perhaps for a chapter on "Permutations and Combinations," the book contains a satisfactory selection of material. There are plenty of illustrative examples throughout the book and numerous exercises to practically all of which answers are given. Historical notes are scattered throughout the text to give some indication of the development of the main notions of the science.

RAYMOND W. BARNARD

Solid Analytical Geometry and Determinants. By Arnold Dresden. John Wiley & Sons, Inc., New York, 1930. x+310 pages. $3.00.

The nature of this book is best explained by quotation from the preface: "Though books on plane analytical geometry frequently devote some chapters to the geometry of a space of three dimensions, the material covered in these chapters is with few exceptions, not intended to do more than provide a general introduction to the subject . . . it rarely goes far enough to acquaint them (the students) with the more interesting and valuable methods of this field . . . it has seemed to the author that, in the study of Solid Analytical Geometry, the young student of mathematics can find an excellent opportunity for an introduction to methods and principles which have an important part in various fields of advanced mathematics. Among these are the methods based on the theory of determinants and on the concept of the rank of a matrix . . . they find relatively simple application in the subject to which this book is devoted. . . . For these reasons the first chapter of this book presents an exposition of some of the properties of determinants and matrices, followed in Chapter II by a treatment of systems of linear equations. . . . With the basis thus provided

it becomes possible to deal with the geometrical questions of the later chapters in a way which lends itself readily to extension to problems of a more general character. . . . Chapters III to X deal with the loci of equations of the first and second degree in three variables from the point of view of real, metric geometry. Elements at infinity and complex elements are considered as non-existent. . . . In subject matter the last eight chapters follow largely the traditional content of introductory courses in Solid Analytical Geometry. . . . The exercises form an integral part of the course which this book presents. . . . A good many of the problems serve no other purpose than that of illustrating the material in the text. But there are other problems . . . which require a certain amount of original thinking."

Professor Dresden has, in our opinion, admirably carried out his purpose as outlined in his preface. The result is a conveniently small, attractive and readable book suitable for undergraduates of fairly high quality. We can however but wonder where such undergraduates are to be found in numbers sufficient to justify the regular giving of the course covered in this book. Doubtless the assimilation of its contents would be of great value to students intending to specialize in mathematics; it contains a large amount of valuable geometrical information with a thorough study and many applications of determinants and the elementary theory of matrices. We feel that the chief emphasis is on and perhaps the author's predilection is for the algebraic method rather than the geometrical fact. This is doubtless as it should be in view of the greater applicability of the former. There are eight appendices, pages 296–301, of which seven concern determinants. Number VI might be included with IV. There is a good index, pages 303–310.

The book is attractively made and is remarkably free of both misprints and errors. In a careful reading we have noted but one misprint, the omission of the exponent γ from the first symbol on page 22; two unimportant errors: the statement of example 21 on page 107 is not true; the incorrect statement on page 141 that conjugate hyperbolas have equal eccentricities. The style of the book is pleasing, the modesty of the author, continually in evidence, is winning. The frequent occurrence of the phrase, "it should be clear" provokes a smile—it is so much nearer the truth than the usual "it is clear." There follow a number of criticisms of minor importance. "It should be clear" that these criticisms are not intended to show the slightest condemnation of the book, which we thoroughly like and approve, but are only in the nature of the duty of a reviewer. The distinction between a determinant and the value of a determinant seems to us finespun and also unfortunate as tending to annoy the beginner; the definition of a determinant as "a square array of numbers to which a single number, called the value of the determinant, is attached" · · · , page 1, is different from the definitions of Weber and Bôcher, both of whom define the determinant as what Dresden calls the expansion of the determinant. Further the definition is too much like that of a matrix, and finally, we do not see that the distinction is

of any use. We deprecate such a colloquial sentence as the following page 19: "In neither case will these operations kill off a non-vanishing minor of a nor bring a vanishing minor back to life." The usual way of giving the derivative of a determinant, $A(t) = |a_{ij}(t)|$, is $A'(t) = \sum a_{ij}'(t) A_{ij}$, where A_{ij} is the cofactor of a_{ij}, which seems to us a more compact and simple form than that given on page 32. It would perhaps be worth while to introduce into the book the idea of continuity, as is done in many books on Algebra. Some proofs might thereby be simplified, others presented which are not here given at all. See, for example, pages 42 and 125. In defining the direction angles of a directed line, page 55, it is necessary only to consider angles from 0° to 180°, not from −180° to +180°, since there is no question of the direction of measuring the angle. The proof of theorem 13, page 63, is far from being the simplest available. The distinction between right and oblique cylindrical surfaces, page 69, is not a distinction between different kinds of surfaces as might be supposed by the student, nor is it necessary to suppose the directrix to be a plane curve. It is not clear what the author means by "unsigned length," page 75, nor does the use of this term seem to be advantageous. In the introduction to Chapter V, "Other Coordinate Systems," page 108, the necessity of a choice of units is emphasized. The emphasis would be more appropriate at the beginning of Chapter III, page 49, where cartesian coordinates are first used. In the definition of spherical coordinates, page 108, the direction of measuring angles should be stated. In studying the shape of a surface, paragraph 70, pages 135–140, no mention is made of the traces of the surface on the coordinate planes. These traces are, we believe, of greater help to the student in visualizing a surface than the projections of the plane sections, which are here called by the author "contour lines," page 138. In the very interesting discussion of focal curves and directrix cylinders of central quadrics, paragraph 120, 121, pages 281–287, the use of the word "any" in definition IV., p. 281, is so confusing as to be actually misleading. It is to be regretted that some of the other interesting and important properties of focal conics are neither given in the text nor incorporated as exercises.

J. K. WHITTEMORE

Notions sur la Géométrie Réglée et sur la théorie du Complexe Quadratique (*Appendice au Cours de Géométrie analytique*). By Georges Bouligand. Paris, Librairie Vuibert, 1929. 84 pages. 11 francs.

The student ignorant of line geometry and lacking the text to which this little pamphlet is an appendix would find the brochure difficult reading. He would first encounter a smooth development of radial and axial coordinates, but would next find a discussion of a complex, and a complex cone, without finding a definition of these concepts. Similar lacunae are plentiful throughout. Frequent references to the *Géométrie Analytique* emphasize the fact that we are dealing with an appendix which the author doubtless did not intend to have stand on its own feet.

In rapid fashion are discussed one- and two-parameter families of linear complexes, Klein's interpretation of line geometry as that of points of a V_4^2 in S_5, six complexes in mutual involution, and the quadratic complex. In a scant eight pages, one finds the topics of special complexes, Chasles correlation, singular congruence, focal and developable surfaces of a congruence, etc. There is an equally brief space devoted to tetrahedral, Battaglini, and Painvin complexes, with indications of the Kummer surface.

Twenty-three pages are devoted to exercises and *questions d'agrégation*.

The reviewer feels the pace is far too swift for the beginner; the advanced student is perhaps surprised to see a reference to Koenigs and Jessop only in the addenda, and none at all to Sturm, Zindler, or Hudson's classic book on the Kummer surface.

C. A. RUPP

An Introduction to the Geometry of n Dimensions. By D. M. Y. Sommerville. E. P. Dutton, New York, 1930. xvii–196 pages.

Although the study of the geometry of n dimensions had its origin almost ninety years ago in a paper by Cayley, it had attracted comparatively little attention until recently. Much of the work so far accomplished in this branch of mathematics has been accomplished by Italian geometers. During recent years in England, the country of its origin, mathematicians have shown increasingly keen interest in the study and a fair number of articles have been written upon various aspects of the subject. Books in English dealing with this field of endeavor are scarce, and not much less so in other languages. An inquiring mind wishing an easy avenue of approach to an understanding of the nature of some of the beautiful hyperspatial configurations and relations in hyperspace is almost completely at sea. In view of this scarcity of books on the subject and also in view of the fact that a knowledge of hyperspace geometry is becoming more and more a necessity to a student of mathematics, the appearance of the book under review is timely.

Mr. Sommerville, the author, is a mathematician of high rank and quality and one is tempted to say that whatever he writes is worth reading. His book, *An Introduction to the Geometry of n Dimensions*, is certainly worth reading, worth studying. It is not a textbook nor a systematic treatise, but it is what it is, an introduction, introducing inquiring students, already having a fair knowledge of ordinary geometry, to the various representative topics of n-dimensional geometry.

The topics are well chosen and they appear in the ten chapters comprising the book as follows: Ch. I, Fundamental Ideas; Ch. II, Parallels; Ch. III, Perpendicularity; Ch. IV, Distances and Angles between Flat Spaces; Ch. V, Analytical Geometry: Projective; Ch. VI, Analytical Geometry; Metrical; Ch. VII, Polytopes; Ch. VIII, Mensuration: Content; Ch. IX, Euler's Theorem; Ch. X, The Regular Polytopes. At the end of every chapter valuable references are given.

All the chapters are well written, and the contents are clearly presented. Elementary and familiar ideas are first carefully explained and emphasized so as to lead the reader gradually to ideas of increasing difficulty. A thoughtful reader, familiar with geometrical reasoning both synthetic and analytic, will find it not too difficult to understand the book.

The book is well printed and is, to the reviewer's knowledge, free of misprints. On page 10, the terms *congruence* and *complex* are used in place of *complex* and *congruence* respectively but on other pages they are correctly used.

One may wish that other topics were included in the book but, the book, as it is, being a brief introduction to n-dimensional geometry, serves its purpose and should be in the hands of all those students of mathematics who have any interest in geometry at all.

B. C. WONG

Johannes Kepler in seinen Briefen. Edited by M. Caspar and Walter Von Dyck. R. Oldenbourg, Munich, 1930. xxvii+396+xvi+348 pages. 8 illustrations.

These two attractive volumes of letters selected from Kepler's correspondence make an important addition to the literature relating to one of Germany's great geniuses that is in published form at the three hundredth anniversary of his death.

While some of Kepler's letters have appeared previously they have been quite incidental to the presentation of some of his theories and have been used in fragmentary form as notes, appendices, and illustrative material. The fact that they were written in Latin or in the awkward ceremonious German of his time has been a serious handicap to their wide circulation. The present editors feel that they have done a favor to Kepler's admirers and all who appreciate contact with such intellectual strength as his in making available in easily readable German this highly valuable material.

The editors indicate that the selection has been made from some four hundred letters with painstaking care and with the purpose of making it of more general interest than is usually associated with Kepler. While technical subjects may have suffered by this treatment, a great deal of light is shed on Kepler's colorful character development from his school days at Tübingen to his death at Regensburg. His letters show in an unusually vivid way the greatness of his thoughts, the depth of his feelings, and the richness of his spiritual life.

Each of the two volumes contains five groups of letters corresponding rather closely to Kepler's residence in different places. The first of the Tübingen student letters is a request in refreshing youthful style for funds. The Grazian group concerns his first marriage, his calendar, which seems to have established his reputation as astrologer, and includes interesting correspondence with Tycho Brahe and Galileo. A group concerning his visit at Prague with Tycho is followed by the set incidental to his connection with the court of Rudolph II and

then by the series in which Galileo's discoveries with the telescope are the chief subject.

The second volume begins with letters on theological questions, his second marriage, and incidents in the development of the witchcraft entanglement of his mother. Kepler's active participation in the suit that followed and the influence on his philosophy are reflected in the set of letters which concludes with his life at Lenz. The Rudolphine Tables, family sorrows, and melancholy forebodings predominate in the 1627–28 correspondence during his Ulm residence. The letters written from Prague and Sagan in the final years of his life are in relation to lectures, affairs of his printing business, and matters incidental to his daughter's marriage. The correspondence closes with two letters written by Kepler within the month previous to his death, in November 1630, and others, between family and friends, written in the two months following.

The volumes are artistically bound with linen and partially covered with bluish-gray paper on which a rough etching of Kepler has been impressed. The printing is in heavy German type on a good grade of paper. The frontispiece of the second volume, a photostat copy of a page of the calendar in Kepler's writing, is especially interesting. The five illustrations in each volume are for the most part portraits of the principal correspondents.

F. E. CARR

PROBLEMS AND SOLUTIONS

EDITED by B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible however, the editors will be glad to assist members of the Association in the solution of such problems.

3489. *Proposed by J. G. Deutsch, Columbia University.*

Given four points in a plane, no three of which are collinear, what is the necessary and sufficient condition that there exist a square containing the given points on its perimeter such that no two points are on the same side?

3490. *Proposed by H. E. Stelson, Kent State College, Kent, Ohio.*

Sanford's *History of Mathematics*, page 264, tells of Pappus' (c. 300 A. D.) method of trisecting an angle by the use of conics. He placed the angle in quest-

ion with its vertex at the center of a circle. He then trisected the subtended chord AB and constructed a hyperbola having AA' two thirds of the chord as its transverse axis and $\sqrt{3}$ times the transverse axis for the conjugate one.

Prove analytically that points joined to the ends of the chord such that one of the angles thus formed is double the other lie on the above hyperbola.

UNSOLVED PROBLEMS

Solutions are desired for the following unsolved problems.

238 [1916, 19]. *Proposed by Clifford N. Mills.*

Determine the rational value of x that will render x^3+px^2+qx+r a perfect cube. Apply the result to $x^3-8x^2+12x-6$.

461 [1916, 209; 1919, 414]. *Proposed by E. T. Bell.*

(1) Two events have probabilities p, q respectively. The events may be either (i) mutually independent; or (ii) mutually exclusive. Assign meanings to the symbol p^q, in terms of the two events where p^q is written for $p\times p\times \cdots \times p$, ($q$ factors p), in cases (i), (ii), and $p\times p$ has the customary meaning (as a probability).

(2) What relations, if any, other than (i) and (ii) can exist between two events? Upon what postulates is the answer to this based?

415, [1916, 301]. *Proposed by George Paaswell.*

If r is the distance from a fixed point (x, y, z) to a variable point (x', y'), in the plane $z=0$, determine the value of the integrals $\int\int r dx'dy'$ and $\int\int \log(z+r)\, dx'dy'$ for the two cases: (a) when the integration is extended over the surface of the circle of radius R; and (b) when the integration is extended over the surface of the rectangle of dimensions a, b.

These integrals are special cases of the direct and logarithmic potentials, the densities of the surface distributions being taken as unity.

343 [1917, 124]. *Proposed by J. Rosenbaum.*

Two bodies of equal masses and coefficients of friction μ_1 and μ_2 are connected by a light spring of stiffness k and placed on an inclined plane. Discuss the motion of each body when the angle between the non-stretched spring and the plane is θ.

344 [1917, 177]. *Proposed by J. Rosenbaum.*

Two bodies of equal masses, and coefficients of friction μ_1 and μ_2 are connected by a light, flexible string, and placed on an inclined plane. What is the angle, θ, between the string and the plane if the inclination, α, of the plane is a minimum when the bodies are on the point of motion?

429[1917, 231]. *Proposed by N. P. Pandya.*

Trace the curve given by the solution of

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=\frac{dy}{dx}+\frac{1-4x^2}{(1-x^2)^{3/2}}.$$

432[1917, 287]. *Proposed by R. P. Baker.*

The expressions

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right)$$

and

$$x^{-(i+1)}\left(x^3\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}\right)$$

are formally equivalent for every integral value of i.

SOLUTIONS

3369[1929, 169]. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two equilateral triangles one within the other, to construct a third equilateral triangle which shall be inscribed in the outer and circumscribed about the inner.

I *Solution by Otto Dunkel, Washington University*

Let ABC and $A'B'C'$ be the outer and inner equilateral triangles; it is required to inscribe in the first an equilateral triangle XYZ which also circumscribes the second. The results and notations in the solution [1931, 229] and discussion of 3441 [1930, 380] will be used in what follows. Since ABC is equilateral, there is no point O', and the point O must be its center and also the center of XYZ. Moreover, the sense of rotation of ABC and XYZ must be the same. Similarly, the centers of XYZ and $A'B'C'$ must coincide. It is supposed that A' lies on YZ, B' on ZX, and C' on XY. Hence the three triangles must be concentric and have the same sense of rotation. So we need to consider in the construction only one vertex of $A'B'C'$, which will be denoted by P.

The three parabolas for ABC are tangent to its sides at its vertices; the vertices of the parabolas are the mid-points of the altitudes of ABC; and the other intersections of the parabolas lie on these altitudes one ninth of their length from their respective bases. The regions I, II_a, II_b, II_c lie within ABC, while the regions IV lie outside. If P lies within I there is no construction; if it lies elsewhere within ABC there may be from one to four constructions.

If P lies within II_c an arc of a circle is described on OP so as to contain 30°, for example on the side of OP towards B. This arc cuts BC in two points, each of which is an X for the required triangle. No other side of ABC will be cut by this arc. If P lies within III_{ca} an arc of a circle is described on OP so as to contain 30°, towards C for example; this arc cuts BC in two points, each of which is

an X for two of the required triangles. The arc cuts also CA in two points, which are the Y's for the remaining two triangles. All of these triangles are inscribed in the strict sense.

Thus, if P is a point within ABC and the arc on OP containing 30° does not cut any side, P lies within I. If it cuts only one side in two points, P is within a II; if it cuts two sides each in two points P is within a III.

3374 [1929, 233]. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two triangles, one within the other, to construct a third triangle which shall be inscribed in the outer and circumscribed about the inner triangle. Also prove that if the two given triangles are equilateral and concentric the third triangle is equilateral.

A Note by Otto Dunkel. The purpose of this note is to indicate the omission of a special case in the solution [1930, 160] of this problem. This special case in the notation of that solution is that in which $C'A'$ passes through C, $A'B'$ through A, $B'C'$ through B. For this situation of the inner triangle there are an infinite number of constructions of the desired triangles. The proof follows the method in the first part of the solution mentioned. Let $C'A'$ cut AB in P_1; $A'B'$ cut BC in Q_1; $B'C'$ cut CA in R_1. Then corresponding to the points P, Q, R, S, we find the three sets of points

$$A, Q_1, A, A; \quad B, B, R_1, B; \quad P_1, C, C, P_1.$$

Thus A, B, P_1 are self-corresponding points in the two projective ranges P and S on AB; and hence every point is self-corresponding; this means that S will always fall upon P. Here $A, B, C; A', B', C'; P, Q, R$ have the same sense of rotation, and the inner triangle is situated in this special position. This is the only way in which there can be an infinite number of constructions. Since this exceptional case has only recently been observed by the writer, it is probable that he also has overlooked it in the application of the related theorem to the solution of previous problems in the Monthly.

Passing now to the second part of the problem we see that the theorem there stated is not true, if we use the terms inscribed and circumscribed in the most general sense, unless we exclude this situation of the inner triangle as well as the case where ABC and $A'B'C'$ have opposite senses. In order to state the results for this second part it will be convenient to change the notation. The required triangle is denoted by XYZ, where X lies on BC, Y on CA, Z on AB; and A' lies on YZ, B' on ZX, C' on AB. The solution of 3369 in this issue shows the number of constructions of equilateral triangles XYZ for the several regions of the interior of ABC. In these constructions the sense of the three triangles is the same. With this requirement of sense fulfilled there are infinitely many triangles XYZ, if a vertex, say C', falls on the arc of a circle (C) through A, B and the center of ABC. We exclude the points A, B, and the center of ABC. In this case two vertices of XYZ lie within or at the ends of the corresponding sides of ABC while the third vertex lies on a prolongation of its corresponding

side. These triangles include special cases where two sides of XYZ are parallel and distinct or coincident. None of these triangles can be equilateral.

If ABC, XYZ have the same sense while $A'B'C'$ has the opposite sense, there cannot be an infinite number of constructions. But to each situation of $A'B'C'$ there are two distinct triangles XYZ; and here again one vertex lies on a prolongation of a side, while the other two lie on their respective sides. None of these triangles are equilateral. Hence there are always six possible constructions, as we see by renaming the points $A'B'C'$ in cyclic order. If a vertex, say C', lies on the circle (C) then one triangle degenerates into a triangle having two sides coincident. If a vertex of $A'B'C'$ falls within one of the III regions, then there are four constructions for equilateral triangles and six for non-equilateral triangles. Since this region does not contain in its interior any point of circles similar to (C) none of these triangles can have two sides coincident. But if a vertex lies on one of two certain hyperbolas passing through the center and a pair of mid-points of the sides of ABC, two sides of one triangle (of the six) are parallel. Hence there are regions for which we can construct ten true triangles, four of which are equilateral. Part of the arc AB of circle (C) lies in I and the other two parts in II_a and II_b, and the total number of constructions may be stated for these regions.

3413 [1930, 157]. *Proposed by P. R. Rider, Washington University.*

The axis of a right circular cylinder of radius r coincides with a diagonal of a unit cube. Find the area of that part of the cylindrical surface which is inside the cube. Partial Solution (1930, 557) by V. F. Ivanoff, San Francisco, California.

II. *Solution by H. L. Rietz, University of Iowa*

Locate the cube with respect to rectangular coördinate axes as shown in the attached figure.

Let $P(x, y, z)$ be a point of the surface of the cylinder with OD as its axis. Then

$$x + y + z = 3m \tag{1}$$

is a plane normal to the axis OD of the cylinder cutting OD at $P_1(m, m, m)$, and

$$r^2 = (x - m)^2 + (y - m)^2 + (z - m)^2. \tag{2}$$

We shall find it convenient to locate P in another way by giving its distance p from the plane through O normal to OD, together with the plane polar coördinates (r, θ), where θ is the angle which P_1P makes with the line P_1N normal to OD and intersecting the z-axis at a point N.

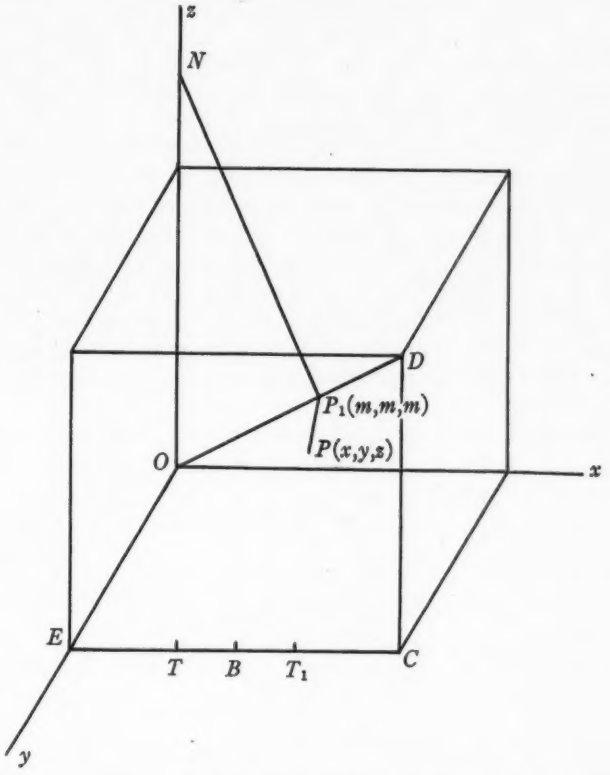
The direction cosines of P_1N being $-6^{-1/2}$, $-6^{-1/2}$, $2\cdot 6^{-1/2}$, and those of P_1P being $(x-m)/r$, $(y-m)/r$, $(z-m)/r$, we obtain

$$\cos\theta = \tfrac{1}{2}r^{-1}\cdot 6^{1/2}(z - m). \tag{3}$$

When $z=0$ in (3), we have $m=-2\cdot 6^{-1/2}r\cos\theta$, and

$$p_1 = 3^{1/2}m = -2^{1/2}r\cos\theta, \tag{4}$$

where p_1 is the lower bound of p for an assigned θ.



To get the upper bound of p in the region[1] of the cube from $\theta=2\pi/3$ to π, we shall find p in terms of r and θ when $y=1$. For this purpose, we first make $y=1$ in equations (1) and (2). Next we solve for $z-m$, and then for m, using the value of $z-m$ from (3).

This gives

$$m = 1 + 6^{-1/2}r\cos\theta \pm 2^{-1/2}r\sin\theta \tag{5}$$

and

$$p_2 = 3^{1/2} + 2^{-1/2}r\cos\theta \pm 3r\cdot 6^{-1/2}\sin\theta, \tag{6}$$

[1] From symmetry this region from $\theta=2\pi/3$ to $\theta=\pi$ is obviously one-sixth of the total region with which we are concerned.

where p_2 is the upper bound of p for an assigned θ. It is easily verified that the negative sign should be used before $3r \cdot 6^{-1/2} \sin \theta$ in (6).

When $r \leqq 2^{-1/2}$, the surface of the cylinder is given by

$$6 \int_{2\pi/3}^{\pi} (p_2 - p_1) r d\theta = 2 \cdot 3^{1/2}(\pi - 3 \cdot 2^{1/2} r) r, \tag{7}$$

which checks with the result found by V. F. Ivanoff.

The foregoing would be unnecessary except as an introduction to the more complicated part of the solution.

For values of r in the remaining interval $2^{-1/2}$ to $2 \cdot 6^{-1/2}$ within the cube, the cylinder will cut the edge $EC (y = 1, z = 0)$, and our integration will be separated into parts. Denote by S_1 the surface of the part for which the intersections of the cylinder with EC are along EB, say at T, the point B being the middle point of EC. Denote by S_2 the surface of the remaining part, that is, the part for which the intersections of the cylinder with EC are along BC, say at T_1.

In the integration to find S_1, we take $2\pi/3$ for the lower limit of θ, and the value of θ at a point T for the upper limit. In the corresponding integration to find S_2 we take the value of θ at a point T_1 for the lower limit and $\theta = \pi$ for the upper limit.

To find the values of θ that correspond to T and T_1, drop a normal[2] TP_2 from T to OD. At its foot, P_2, draw a normal P_2M where M is on OC. Then it follows by very simple geometry that

$$-3^{1/2} r \cos \theta = 2^{1/2} - r \sin \theta. \tag{8}$$

Solving (8) for $\sin \theta$ and $\cos \theta$, we obtain

$$\sin \theta = \tfrac{1}{4} r^{-1} \cdot 2^{1/2} \pm \tfrac{1}{4} r^{-1} (12r^2 - 6)^{1/2}, \tag{9}$$

$$\begin{aligned} \cos \theta = -\tfrac{1}{4} r^{-1} \cdot 6^{1/2} \pm \tfrac{1}{4} r^{-1} (4r^2 - 2)^{1/2} &= \lambda, \text{ when } + \text{ sign is used,} \\ &= \lambda_1, \text{ when } - \text{ sign is used.} \end{aligned} \tag{10}$$

The λ applies to θ corresponding to a point such as T on the left of the middle, B, of EC while λ_1 applies to θ for a point such as T_1 on the right of B. Moreover, it is easy to verify that the positive sign in (10) corresponds to the positive sign in (9). Now we may write

$$S_1 = 6 \int_{2\pi/3}^{\arccos \lambda} (p_2 - p_1) r d\theta = 6r \left\{ 3^{1/2} \arccos \left[-\tfrac{1}{4} r^{-1} \cdot 6^{1/2} + \tfrac{1}{4} r^{-1} (4r^2 - 2)^{1/2} \right] + (6r^2 - 3)^{1/2} - 2\pi \cdot 3^{-1/2} - 3r \cdot 6^{-1/2} \right\}. \tag{11}$$

$$S_2 = 6 \int_{\arccos \lambda_1}^{\pi} (p_2 - p_1) r d\theta = 6r \left\{ 3^{1/2} \pi - 3r \cdot 6^{-1/2} - 3^{1/2} \arccos \left[-\tfrac{1}{4} r^{-1} \cdot 6^{1/2} - \tfrac{1}{4} r^{-1} (4r^2 - 2)^{1/2} \right] + (6r^2 - 3)^{1/2} \right\}. \tag{12}$$

[2] Not shown in the figure.

From (11) and (12), we obtain

$$S_1 + S_2 = 6r[-3^{1/2}\arccos(r^{-2}-1) + 2(6r^2-3)^{1/2} + 3^{-1/2}\pi - 6^{1/2}r]$$

for the surface of the cylinder when

$$2^{-1/2} \leqq r \leqq 2\cdot 6^{-1/2}.$$

Also solved by J. B. Reynolds.

A Note by Otto Dunkel. The analysis given in the partial solution of this problem [1930, 557] may be extended without difficulty to cover the remaining case. It seems quite likely that the solver was aware of this simple extension but did not give the result on account of its complicated form. Since this remaining part has appeared of interest, the rest of the analysis is given below using the numbered results in the reference above. The lateral area of that part of the cylinder outside the cube will be computed. The projection on the xy-plane of that half of a face considered in the solution is an equilateral triangle OAB with an edge of length $2^{1/2}3^{-1/2}$ and an altitude of length $2^{-1/2}$. Here the circle of radius r cuts AB in two points; let us call 2ψ the angle between the two radii to these points. Then $\cos\psi = 2^{-1/2}r^{-1}$, and the angles for these two radii are $\theta_2 = \frac{1}{6}\pi + \psi$, $\theta_1 = \frac{1}{6}\pi - \psi$. Inserting these limits in the integral in (5), and using the reduction $\sin\theta_2 - \sin\theta_1 = 3^{1/2}\sin\psi$, there results

$$12\cdot 3^{1/2}r[\arccos(2^{-1/2}r^{-1}) - (2r^2-1)^{1/2}], \tag{6}$$

$$2^{-1/2} \leqq r \leqq 2^{1/2}3^{-1/2},\quad 0 \leqq \arccos(2^{-1/2}r^{-1}) \leqq \tfrac{1}{6}\pi.$$

If the result in (6) is subtracted from (5) the difference is that part of the lateral area of the cylinder which is inside the cube.

3439 [1930, 315]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

In the *Pecqueur Bibasal Sextic Spur-Gear Train*, one of the 105 angular velocity ratios which are associated with it is

$$N_{31}/N_{61} = (z_2z_4z_5z_6)/(z_3z_4z_1'z_5' - z_3z_4z_5z_6 - z_2z_3'z_1'z_5'), \tag{1}$$

in which the eight z's designate the numbers of teeth on the eight gears of the train. In designing such a train the z's must be chosen in accordance with the following fundamental requirements:

(A) All of the z's must be positive integers.

(B) No z may be less than a certain minimum, say 12, nor greater than a certain maximum, say 60.

Four questions now present themselves, in answering which it must be remembered that whatever special conditions may be imposed on equation (1), general conditions (A) and (B) must always be satisfied. The four questions are:

(I) Between what limits lie the possible numerical values of the ratio N_{31}/N_{61}?

(II) For an assigned, allowable value of the ratio N_{31}/N_{61},—say for the value 1,000,000, if that should prove to be an allowable value,—select the z's.

(III) If possible, so select the z's that the special condition

$$N_{31}/N_{61} = \pm z_2z_4z_5z_6$$

will be satisfied.

(IV) If possible, so select the z's that the following special requirements will be simultaneously fulfilled: (*a*) the ratio N_{31}/N_{61} is to have an assigned numerical value,—say 3,373,785 ($=1809\times1865$) if that should prove to be an allowable value; (*b*) the sum $z_2+z_3+z_3'+z_4+z_4'+z_5+z_5'+z_6$ is to be a minimum.

Solution by Howard H. Mitchell, University of Pennsylvania

A slightly modified form of this problem was sent to me some time ago by Professor Louis O'Shaughnessy, a colleague of Professor Rasche, so that I had some familiarity with it before it was proposed for solution in the Monthly. While any method of solution would seem to involve a certain amount of trial, the following is suggested as possibly reducing that element to a minimum.

As the numerator of the given fraction is a product of four numbers each $\leqq 60$, it is a product of primes <60. If we restrict attention to possible solutions for which it assumes values between the limits $1{,}000{,}000\pm100$, and divide 1,000,100 by suitable powers of the primes less than 60, we find the prime factors that are less than 60 of the numbers between these limits. The only such numbers that have no larger prime factors are $999{,}925=5^2\cdot23\cdot37\cdot47$; $999{,}936=2^9\cdot3^2\cdot7\cdot31$; $999{,}949=29^3\cdot41$; $999{,}999=3^3\cdot7\cdot11\cdot13\cdot37$; $1{,}000{,}000=2^6\cdot5^6$; $1{,}000{,}008=2^3\cdot3^2\cdot17\cdot19\cdot43$; $1{,}000{,}065=3\cdot5\cdot11^2\cdot19\cdot29$.

For each of these numbers there are possible choices for z_2, z_4, z_5, z_6. In order that the denominator of the given fraction shall have the value ±1, the values assigned to z_2, z_4 can have no common factor. There are thus only two possible ways of choosing these four numbers if their product is to be 999,949.

For a particular selection of these four numbers we then seek possible values of z_3, z_3', z_4', z_5' that satisfy the equation

$$z_3z_4z_4'z_5' - z_3z_4z_5z_6 - z_2z_3'z_4'z_5' = \pm 1. \tag{1}$$

The possible values of z_4', z_5', in addition to being between 12 and 60, must evidently be prime to z_4, z_5, z_6. For values of the variables that satisfy (1), the coefficient of z_3 must be positive, and hence, if z_3 were replaced by 60, and z_3' by 12, the left member would be increased. Hence

$$z_4'z_5' > 5z_4z_5z_6/(5z_4 - z_2). \tag{2}$$

In view of the symmetry of (1) in z_4', z_5' we may assume that $z_4' \leqq z_5'$ and hence that z_5' must be greater than the square root of the right member of (2), while z_4' must exceed the right member divided by 60.

We observe also that the two congruences,

$$z_3z_4z_5z_6 \equiv \mp 1, \text{ mod } z_4'z_5', \tag{3}$$

$$z_3z_4z_4'z_5' \equiv z_3z_4z_5z_6 \pm 1, \text{ mod } z_2, \tag{4}$$

must each be satisfied, where the upper or the lower signs must be used in both cases. A table may then be constructed by use of (3) giving the possible values of z_3 for each of the possible values of z_4' or z_5'. For each possible value of z_5' the corresponding values of z_3 may be read from this table and then z_4' calculated from (4). By reference to the table it will then be found that (3) is not usually satisfied. If, however, (3) is satisfied, then z_3', as calculated from (1), may be an integer (*will* be, if the moduli of (3) and (4) are relatively prime), and, if within the prescribed limits, a solution of the desired sort will be obtained.

Some modification of the procedure may be desirable in case z_2, z_4', z_5' have factors in common. Thus, if z_2 and z_5' have a common factor, it may be desirable, after the calculation of z_3, to divide equation (1) by z_5' before transforming it into a congruence, mod. z_2.

For $z_2\ z_4\ z_5\ z_6 = 999{,}949$, solutions of the desired sort were found to exist. If we take $z_2,\ z_4,\ z_5,\ z_6 = 29,\ 41,\ 29,\ 29$, respectively, the inequality (2) gives $z_4'z_5' > 979$, whence $z_5' > 31$, $z_4' > 16$. The congruences (3), (4) reduce to

$$34{,}481z_3 \equiv \mp 1, \text{ mod } z_4'z_5';$$

$$z_3z_4'z_5' \equiv \mp 12, \text{ mod } 29.$$

For $z_5' = 35$ we find, in the case of the lower sign, $z_3 \equiv 6$, mod. 35, whence $z_3 = 41$. For this z_3, we find $z_4' \equiv 5$, mod. 29, whence $z_4' = 34$. For this z_4', the first congruence is found to be satisfied, and from equation (1) it is found that $z_3' = 17$.

In a similar manner, it is found that a solution is given by $z_5' = 51$, $z_3 = 41$, $z_4' = 33$, $z_3' = 29$.

These two solutions together with those obtained from them by interchanging z_4' and z_5' were all that were found to exist for $z_2,\ z_4,\ z_5,\ z_6 = 29,\ 41,\ 29,\ 29$. No solutions were found when the values of z_2 and z_4 were interchanged.

For each other possible value of $z_2z_4z_5z_6$ solutions could be sought in a similar manner. In general, however, the number of possible sets of values for these four variables would be considerably greater than in the case considered.

3444 [1930, 380]. *Proposed by Frank Morley, Johns Hopkins University.*

In an inversive plane, the general self-conjugate equation, $f(x, \bar{x}) = 0$, of degree three in x and $\bar{x}$ defines a bi-cubic curve, c. Since any circle has with such a curve six common points (intersections or common unique pairs) there are contact circles, touching thrice. There are, it is known, 120 contact circles. If we take three of these, the 9 points of contact either lie on a biquadratic b, or they do not. When they do, the circles are tied (or syzygetic); and the curve, b, meets c in the points of contact of a fourth circle, so that the circles are tied in sets of four. Prove that the four circles of a set touch a circle.

Solution by the Proposer

If a_1, a_2, a_3, a_4 be a set of contact circles of c, the 12 points of contact are on a curve b. We have then $a_1a_2a_3a_4+\mu b^2$ containing c as a factor. The other factor must be another circle a. Thus $a_1a_2a_3a_4+\mu b^2=ca$; whence, a will touch a_i where b meets it again.

The bi-cubic is an alternative to the space sextic of genus 4, according as one wishes to view things inversively or projectively. For the underlying theta-theory see Professor Coble's *Colloquium Lectures*.

3449 [1930, 447]. *Proposed by W. E. Buker, Leetsdale, Pa.*

(1) Find a number X such that X^2+5 and X^2-5 are each square numbers.

Note: This problem was proposed by John of Palermo and solved by Leonardo of Pisa about 1220 A.D. A solution is $X=41/12$; for $(41/12)^2+5=(49/12)^2$; $(41/12)^2-5=(31/12)^2$. How did he arrive at his result?

Reference: Florian Cajori, *History of Mathematics*, pp. 124.

Solution by J. D. Hill, University of California at Los Angeles

By subtracting the two equations $x^2+5=Q^2$ and $x^2-5=P^2$, there results $Q^2-P^2=(Q-P)\ (Q+P)=10$. If a solution exists, we may denote $Q-P$ by y and obtain the system

$$Q-P=y,\ Q+P=10/y.$$

The solution for P or Q, with the original relations, gives $x=(100+y^4)^{1/2}/2y$. Since y is assumed rational, we may write $y=a/b$ where a and b are integers, and we have $x=(100b^4+a^4)^{1/2}/2ab$. The problem now reduces to making the radicand a square number, which can be accomplished by Lagrange's method[1] and an infinite number of solutions obtained, provided (as shown in the reference) we can first find two integers A and B such that $100B^4+A^4$ is a square. In order to obtain this first solution we note that $100+y^4$ will be a square when[2] $y^2=-20t/(t^2-1)$, where t is a rational parameter. For y to be rational, it is sufficient that both numerator and denominator be squares, which is true for the former if $t=-m^2/20$, and for the latter if $t=(-1-n^2)/2n$ where m and n are arbitrary rational numbers. Equating these two values of t and solving for n in terms of m we obtain, $n=[m^2\pm(m^4-400)^{1/2}]/20$. For all values of m which make (m^4-400) a square, n and consequently y will be rational, and thus solutions are obtained. By inspection $m=5$ fulfills the condition and yields $y=20/3$ which gives $x=41/12$ or the solution obtained by Leonardo of Pisa. Placing $a=20$, $b=3$ we may now follow Lagrange's method and obtain as many solutions as desired.

Disregarding Lagrange's method, however, we may obtain a solution different from 41/12 as follows by considering m^4-400. Place $m^2+20=S^2$, m^2-20

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[1] See Dickson, *History of Theory of Numbers*, vol. II, p. 628.

[2] See Wertheim, *Zahlentheorie*, p. 56.

$=R^2$ and subtract, obtaining $S^2-R^2=(S-R)\ (S+R)=40$. If we denote $(S-R)$ by z we have the equations

$$S - R = z,\ S + R = 40/z.$$

The solution for S or R taken with our original relations gives $m=(z^4+1600)^{1/2}/2z$. By inspection, $z=3$ yields a rational value of m, namely $41/6$. For this value of m, $y=4920/1519$ which when substituted above gives $x=3{,}344{,}161/1{,}494{,}696$. The two resulting squares turn out to be

$$x^2 - 5 = (113,\ 279/1,\ 494,\ 696)^2$$

and

$$x^2 + 5 = (4,\ 728,\ 001/1,\ 494,\ 696)^2$$

The numerators of the last three fractions are relatively prime to their respective denominators.

This investigation, while not complete by any means, leads directly to Leonardo's solution and to as many others as desired. In the meantime certain leads suggested in the paper are being followed in hopes of securing a more general procedure.

Also solved by Mannis Charosh and J. Rosenbaum.

A Note by Otto Dunkel. The problem asks how Leonardo of Pisa arrived at his result. Since the method above as far as the derivation of the equation $a^4+100b^4=u^2$, where a, b, u are integers, would occur to anyone trying to solve this problem, it would require no great strain on the imagination to suppose that Leonardo proceeded in this way. A very few trials with the first integers would have led him to the solution $a=3$, $b=2$, and then to his result $x=41/12$.

It may be of interest to note that the method of Lagrange, alluded to above, which tells how to get a second solution a_1, b_1 from a first solution a, b leads precisely to the second solution above, if we start with $a=3$, $b=2$. For in that method $a_1=a^4-100b^4=-1519$, $b_1=2abu=492$. From these values of a_1 and b_1 follows the value of x given above.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Franklin Institute, Philadelphia, has awarded a Franklin medal to Sir James Jeans, " in recognition of his many fruitful contributions to mathematical physics, especially in the realms of dynamical theory of gases and the theory of radiation, and of his challenging explanations of astronomical problems and his illuminating expositions of modern scientific ideas."

The New York Academy of Sciences has awarded one of its A. Cressy Morrison prizes to Professor H. von Zeipel, of the Astronomical Observatory of Upsala, for his paper entitled *The evolution and constitution of stars.*

Professor George David Birkhoff, of Harvard University, will give a series of lectures on "Le dernièr théorème de géométrie de Poincaré, ses généralisations, et ses applications à la dynamique," at the Collège de France, during the last two weeks of April, under the Fondation Michonis.

Professor E. B. Stouffer has been elected a member of the section committee of Section A of the American Association for the Advancement of Science.

Dr. Nola L. Anderson has been appointed associate professor and acting chairman of the department of mathematics at Sophie Newcomb College.

Dr. John Jay Gergen and Dr. Albert Eldred Currier have been appointed Benjamin Peirce Instructors at Harvard University for the next academic year.

Associate Professor George W. Hess, of Howard College, has been promoted to a professorship of mathematics.

Dr. H. K. Hughes, formerly instructor at the University of Michigan, has been appointed assistant professor of mathematics at Purdue University.

Dr. Wolfgang Pauli, of the Zurich Technical School, and Dr. Arnold Sommerfeld, of the University of Munich, have been appointed lecturers in theoretical physics at the University of Michigan for the summer session of 1931.

President A. E. Whitford, of Milton College, Milton, Wisconsin, who resigned his office last June, is a lecturer in mathematics at the University of Wisconsin.

Professor David Vernon Widder, of Bryn Mawr College, has been appointed assistant professor of mathematics at Harvard University.

The following courses in mathematics are announced for the summer of 1931:

University of California at Los Angeles, July 1–August 10. By Professor Harriet E. Glazier: Foundations of arithmetic. By Professor E. R. Hedrick: The teaching of mathematics.

The following appointments to instructorships in mathematics are announced:

Columbia University, Mr. E. R. Lorch;

Harvard University, Douglas Payne Adams, Allen Emil Anderson, James Sutherland Frame, Alan Stuart Galbraith, Lester Turner Moston, Sumner Byron Myers, Griffith Baley Price, Frederick Henry Steen, George Booth Van Schaack.

Professor W. A. Zehring, of Purdue University, died recently at the age of 55 years. He was a charter member of the Mathematical Association of America.

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Cohen, Abraham, An Introduction to the Lie Theory of One-Parameter Groups with application to the solution of Differential Equations (1911) Cloth. Reprint 1931....$2.00
F. Klein's Famous Problems of elementary Geometry, ed. Beman & Smith, rev. by R. C. Archibald. 1930. Cloth. XI+92 pages.....$1.50

Of A. G. Webster "Partial Differential Equations of Mathematical Physics" a new edition is in preparation and will be ready within about a year. A German translation with revision by Dr. G. Szegö of Koenigsberg is now available at $7.00.

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By N. J. LENNES
Professor of Mathematics, University of Montana
With the Editorial Cooperation of
H. E. SLAUGHT
Professor of Mathematics, University of Chicago

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BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fifteenth Summer Meeting of the Association, Minneapolis, Minnesota, Sept. 7-8, 1931.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1931.

ILLINOIS, Peoria, May 1-2.
INDIANA, Muncie, May 1-2.
IOWA, Davenport, May 1-2.
KANSAS, Topeka, Jan. 24.
KENTUCKY, Lexington, May 9.
LOUISIANA-MISSISSIPPI, Natchitoches, La., March 13-14.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 9.
MICHIGAN, Ann Arbor, March 21.
MINNESOTA, St. John's University, Collegeville, May 16.
MISSOURI, St. Louis, November.
NEBRASKA, Lincoln, May.
OHIO, Columbus, April 2.
PHILADELPHIA, Philadelphia, Nov. 28.
ROCKY MOUNTAIN, Boulder, Colo., April 17-18.
SOUTHEASTERN, Auburn, Ala., April 24-25.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 21.
TEXAS, Fort Worth, Jan. 31.

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The following thirty-eight persons and one institution have been elected to membership in the Association on applications duly certified:

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A. H. Bailey, B.S. (Wilmington Coll., Ohio). Grad. Student, Haverford College, Haverford, Pa.

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Elizabeth N. Boyd, A.M. (Columbia). Instr., Math. and Chem., Greenbrier Coll., Lewisburg, W. Va.

T. J. Byrnes, B.S. (St. Bonaventure). Teacher, Rochester Shop School, Rochester, N. Y.

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E. H. Cutler, Ph.D. (Harvard). Instr., Lehigh Univ., Bethlehem, Pa.

Grace Erwin, A.B. (Nebraska). Teacher, High School and Normal, Elgin, Ill.

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L. S. Kennison, A.M. (Brown). Research Asst., Princeton Univ., Princeton, N. J.

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C. M. Lennahan, Observer, U. S. Weather Bureau, Ellendale, N. Dak.

N. H. McCoy, Ph.D. (Iowa). Natl. Research Fellow, Princeton Univ., Princeton, N. J.

J. O. Monasterio, C.E. (Chapultepec Milit. Acad.). Math. and Physics, Loyola Univ., New Orleans, La.

Ethel I. Moody, Ph.D. (Cornell). Instr., Sweet Briar Coll., Sweet Briar, Va.

Nellie C. Morton, A.B. (Brown). Asst., Brown Univ., Providence, R. I.

G. A. Newton, A.M. (Trinity). Head of Dept., Trinity Univ., Waxahachie, Tex.

Mabel R. Oldham, A.M. (Texas). Asst. Prof., State Coll. for Women, Denton, Tex.

Arthur Ollivier, M.S. (Iowa). Instr., State Agric. Coll., Manhattan, Kans.

P. S. Pretz, A.M. (St. John's Univ.). St. Benedict's Coll., Atchison, Kans.

Tibor Rado, Ph.D. (Szeged, Hungary). Prof., Ohio State Univ., Columbus, Ohio.

M. C. Rhodes, Ph.D. (Peabody). Head of Dept., Univ. of Mississippi, University, Miss.

A. J. Robinson, M.S. (Emory). Asst. Prof., Alabama Polytech. Inst., Auburn, Ala.

H. D. Ruderman, Undergraduate, Coll. of the City of New York.

N. L. Schultz, Student Instr., Univ. of Maine, Orono, Me.

Wladimir Seidel, Ph. D. (Munich). Natl. Research Fellow, Harvard Univ., Cambridge, Mass.

Edmond Siroky, M.S. (Washington Univ.). Asso. Prof., Washington Univ., St. Louis, Mo.

I. S. Sokolnikoff, Ph.D. (Wisconsin). Asst. Prof., Univ. of Wisconsin, Madison, Wis.

Anna A. Stafford, A.B. (Western Coll.). Teacher, St. John Baptist School, Mendham, N. J.

Austin L. Starrett, A.B., Grad. Student, Harvard University.

J. K. Stewart, M.S. (West Virginia). Grad. Asst., West Virginia Univ., Morgantown, W. Va.

Alexander Tartler, A.B. (Pennsylvania). Instr., Drexel Inst., Philadelphia, Pa.

Ethel Ruth Ward, A.M. (North Carolina). Instr., Math. and Chem., State Coll. for Women, Columbus, Miss.

Clement Weinstein, A.M. (Pennsylvania). Instr., Univ. of Pennsylvania, Philadelphia, Pa.

Samuel Welkowitz, B.S. (C.C.N.Y.). Asst. in Math. F. K. Lane High School, Brooklyn, N. Y.

Charles Wexler, Ph.D. (Harvard). Head of Dept., State Teachers Coll., Tempe, Ariz.

D. V. Widder, Ph.D. (Harvard). Prof., Bryn Mawr Coll., Bryn Mawr, Pa.

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